Chapter 0: Introduction

The Fundamentals of Mathematics – an Overview

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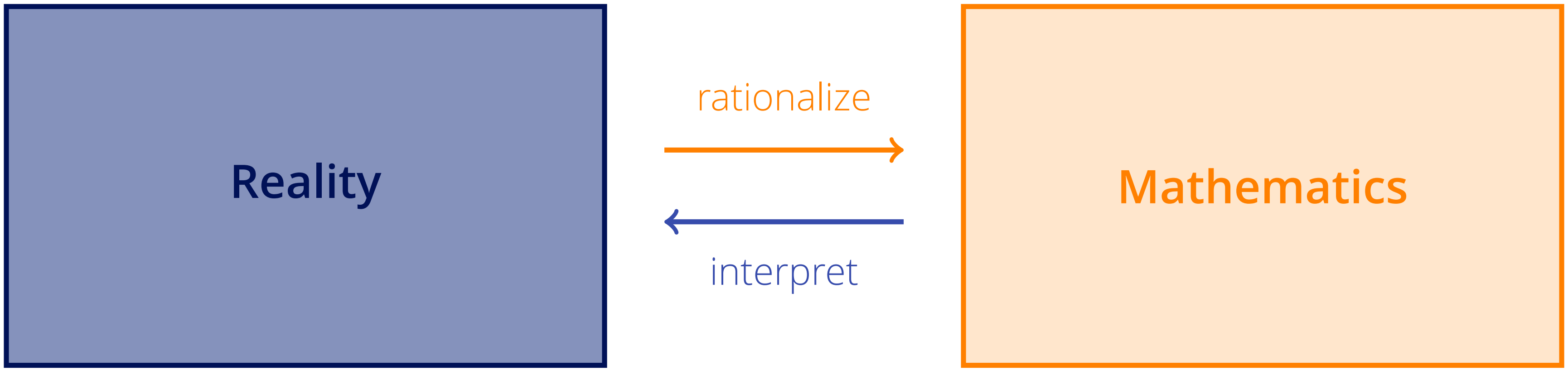
UNIVERSITÄT MANNHEIM

Outline

In this chapter, we review

- mathematical notation and the basics of formal logic
- fundamental definitions of set theory
- terminology and fundamentals of functions
- basic definitions of convergence and continuity
- archetypes of proofs

Why Mathematics?



- Every field of modern economics relies on math
  - Dynamic programming in Macro
  - Structural models in IO/Micro
  - Estimators and their (asymptotic) behavior in Econometrics

- Mathematics formalize and simplify logical reasoning

Mathematical Language

- Math has its own language $\rightarrow$ similar to any "regular" language
- Quantifiers:
  - $\forall$ Universal quantifier (read "for all...")
  - $\exists$ Existential quantifier (read "there exists...")

    $\Rightarrow \nexists$ Non-existence quantifier (read "there doesn't exist...")

    $\Rightarrow \exists!$ Unique esitence quantifier (read "there exists exactly one...")
- Logic operators:
  - $\wedge$ logical *and*
  - $\vee$ logical *or*
  - $\neg$ negation
  - $\Rightarrow$ implication ($A \Rightarrow B$ if whenever A is fulfilled, so is B)
  - $\Leftrightarrow$ equivalence ($A \Leftrightarrow B$ if $A \Rightarrow B$ as well as $B \Rightarrow A$)

$x > 5 \Leftrightarrow x \in (5, \infty)$

$x > 5 \Rightarrow x \in (5, \infty)$

$\wedge \quad x \in (5, \infty) \Rightarrow x > 5$

Arguments & Negation

- Negating statements important for proofs (proving statement $\iff$ disproving negation)

  $\Rightarrow$ logical negation has its own rules $\neg\big(\underbrace{\forall x \in \mathbb{N}}_{\exists x \in \mathbb{N}} \quad \underbrace{x \geq 0,}_{x < 0}\big)$

Rules for Negating Logical Statements

Given a logical statement, the following hold true for its negation

- All quantifiers are indiviudally reversed (same order!)
- The statement's property is negated
- De Morgan's law applies to composite statements

  $\neg(A \wedge B) \equiv \neg A \vee \neg B$ and $\neg(A \vee B) \equiv \neg A \wedge \neg B$
- For an implication, $\neg(A \Rightarrow B) \equiv A \wedge \neg B$ contrapositive

$\neg(\forall x \in \mathbb{N} \quad \exists y \in \mathbb{N} \quad y > x)$

$\exists x \in \mathbb{N} \quad \forall y \in \mathbb{N} \quad y \leq x$

$\neg(\forall x \in \mathbb{N} \quad (x \text{ odd}) \vee (x \text{ even}))$

$\neg(x \text{ odd}) \wedge \neg(x \text{ even})$

$\exists x \in \mathbb{N} \quad x \text{ not odd} \wedge x \text{ not even}$

Set Theory

$\{1, \pi\} = \{\pi, 1\} = \{1, 1, \pi\}$

- Set = collection of distinct objects ("elements")
  - $\{1, 2, \pi\}$ vs. $\{1, 1, \pi\}$ vs. $\{n \in \mathbb{N} : n > 10\}$ $\quad \subsetneq$
  - Elements can be everything (sets, matrices, functions, mixed onjects...)
  - No order, no repetitions $\quad \mathbb{N} \quad \{n \in \mathbb{N} : n > 10\} \to \{n \in \mathbb{N} : n \leq 10\}$
- Set relations
  - superset $\supset$
  - subsets $A \subset B \iff x \in A \Rightarrow x \in B$ (careful about equality, either $\subseteq, \subset$ or $\subset, \subsetneq$)
  - disjoint sets $A, B \iff A \cap B = \emptyset$ $\to$ no common elements
  - complement (with respect to X) $A^c = X \setminus A$ (sometimes also $\bar{A}$)
- Operations A B A or B $\qquad$ A B A and B $\qquad$ A B
  - Union $A \cup B$, Intersection $A \cap B$, Set difference $A \setminus B$ (circles)
  - Power set $\mathcal{P}(A) = \{S \subseteq X : S \subseteq A\}$: set of subsets (sometimes also 2^A)
  - Index sets for *sets/sequences of sets*: $\bigcup_{i \in I} A_i$, $\bigcap_{i \in I} A_i$ (e.g. $I = \mathbb{N}$)
- Cardinality $|A|$ (number of elements): finite, countable, uncountably infinite

$\mathcal{P}\{1, 2, \pi\} = \{\emptyset, \{1\}, \{2\}, \{\pi\}, \{1, 2\}, \{1, \pi\}, \{2, \pi\}, \{1, 2, \pi\}\}$

8 elements in $\mathcal{P}\{\}$

Particular Sets

countable infinity: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$

uncountable infinity: $\mathbb{R}$, $\mathbb{R}^k$

- Natural numbers

  $\mathbb{N} := \{1, 2, 3, \ldots\}$
- Whole numbers

  $\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Rational numbers

  $\mathbb{Q} := \{\frac{p}{q} : p, q \in \mathbb{N}\}$
- Real numbers

  $\mathbb{R}$: "all" one-dimensional numbers
- Positive/Strictly positive reals

  $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ $\mathbb{R}_0^+ := \{x \in \mathbb{R} : x \geq 0\}$ [alternative: $\mathbb{R}_*^+/\mathbb{R}^+$ or $\mathbb{R}_>/\mathbb{R}_\geq$]
- Intervals (in $\mathbb{R}$)

  Open interval (a, b) Closed interval $[a, b]$

$x > 5 \rightarrow (5, \infty)$

$x \geq 5 \rightarrow [5, \infty)$

$[10, 20)$

Functions

$f(x) = x^2$ $\quad f(\mathbb{R}) = \mathbb{R}_0^+$

$f: \mathbb{R} \mapsto \mathbb{R};\ x \mapsto x^2$

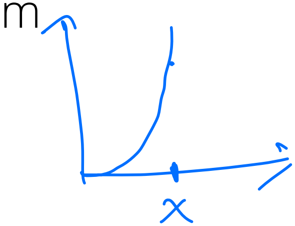
$\mathbb{R} \mapsto \mathbb{R}_0^+$

- Functions define a relation between sets
  - Usual notation: $f: X \mapsto Y, x \mapsto y = f(x)$
    - X : domain, Y : codomain
    - Mapping rule "$x \mapsto y = f(x)$" relates each $x \in X$ to exactly one $y \in Y$
  - Relation = collection of pairs $(x, y) = (x, f(x))$ in $X \times Y$:

(x, x^2)

$$G(f) := \{(x, y) \in X \times Y : y = f(x)\} = \{(x, f(x)) : x \in X\}$$

  - The *set* $G(f)$ is called the graph of f $\rightarrow$ curve drawn in a coordinate system



  - The image $f(A)$ of set $A \subseteq X$: $f(A) := \{y \in Y : \exists x \in A, y = f(x)\}$

$\rightarrow f(2) = 4 \qquad f(\{1,2,3\}) = \{1,4,9\}$

- Related concept: Correspondences [not too relevant for this course]
  - Sometimes also called set-valued functions $\rightarrow$ max, min
  - Relates each x in the domain to a set of elements (instead of just one)

Limits and Continuity in $\mathbb{R}$

$X_n = \frac{1}{n}$

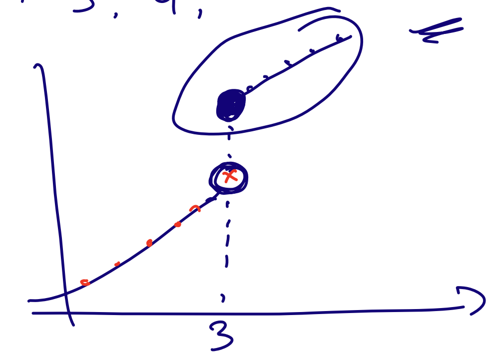
$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \to 0$

- Sequence: $\lim_{n\to\infty} x_n = x \Leftrightarrow |x_n - x| \xrightarrow{n\to\infty} 0$

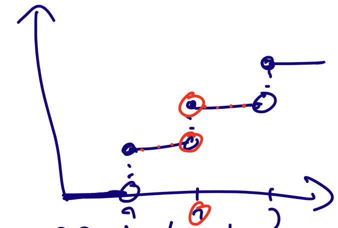
  $\forall \varepsilon > 0 \;\; \exists N : \forall n > N \;\; |x_n - x| < \varepsilon$

- Function $f : X \mapsto \mathbb{R}$ where $X \subseteq \mathbb{R}$:

$$\lim_{x\to a} f(x) = f_a \in \mathbb{R} \;\;\Leftrightarrow\;\; |f(x) - f_a| \underset{x\neq a}{\overset{x\to a}{\to}} 0$$



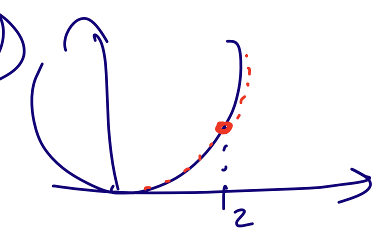
- Continuity of f at $a \in X$: $\lim_{x\to a} f(x) = f(a)$
  - Examples for discontinuity: $f(x) = \lfloor x \rfloor$, $\;f(x) = \mathcal{I}[x = 0]$
  - Characterization: left and right limit exist at a and are equal to $f(a)$



$f(x) = \lfloor x \rfloor$

$$\forall \epsilon > 0 \, \exists \delta > 0 : (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$$

  - Disprove: left $\neq$ right limit, sequence $a_n \to a$ where $f(a_n) \not\to f(a)$



Proofs and Proof Types

$C \Rightarrow D$

$\neg(A \Rightarrow B)$ $\qquad \neg B \Rightarrow \neg A$

- Proofs are a fundamental part of mathematics (and axiomatic Economics)

  $\Rightarrow$ goal: show validity of claim starting from a set of premises (axioms)

- **Direct proof**: Manipulating the axioms until obtaining the claim

  $\Rightarrow$ Example: n^2 has the same parity as n

  $\rightarrow$ Proof by contrapositi

- **Proof by contradiction**: Assuming the negation and showing a contradiction

  $\Rightarrow$ Example: There is no largest prime p^{max}

- **Proof by (complete) induction** Generalizing from simple case to (countable) set
  - Induction base: Showing that claim holds for specific value (e.g. $n = 1$)
  - Induction step: Showing that if claim is true for n it also holds for $n+1$

  $\Rightarrow$ Example: For any $n \in \mathbb{N}$: $\sum_1^n x = \frac{n(n+1)}{2}$

  Axioms $\qquad a = b$, $b = a$

- Claims usually classified into Lemmas, Propositions, Theorems and Corollaries

Proof Types – Examples

- Direct proof: n^2 has the same parity as n

  $n = 2k + i, \;\; i \in \{0,1\} \Rightarrow n^2 = (2k+i)^2 = 4k^2 + 4ki + i^2 = 2\overbrace{(2k^2 + 2ki)}^{=:\tilde{k}\in\mathbb{N}} + i^2$

  Note for $i \in \{0,1\} : i^2 = i \quad \Rightarrow n^2 =: 2\tilde{k} + i$ same parity as $n = 2k + i$

- Proof by contradiction: There is no largest prime p^{max}

  $\Rightarrow$ Note: If n divisible by $k > 1$, then $n - 1$ not divisible by k

  Suppose $\exists p^{\text{max}}$ s.t. $p_i < p^{\text{max}}$ for every prime $p_i \;\; \rightarrow \exists \mathcal{P} := \{2, 3, \ldots, p^{\text{max}}\}$

  Define $p^* := \Pi_{\mathcal{P}}\, p \; - 1 > p^{\text{max}} \quad \rightarrow \quad p^*$ not divisible by any $p \in \mathcal{P}$

  p^* either prime [$p^* > p^{\text{max}}$ $\perp$] or has a prime factor $p' \notin \mathcal{P}$ [$\mathcal{P}$ incomplete $\perp$]

- Proof by (complete) induction: For any $n \in \mathbb{N}$: $\sum_1^n x = \frac{n(n+1)}{2}$

  1. Induction base: For $n = 1$: $\sum_1^1 x = 1 = \frac{1 \cdot 2}{2}$
  2. Induction step: $\sum_1^{n+1} x = \sum_1^n x + n+1 \overset{\text{holds for } n}{=\!=} \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$

n even/odd $\Rightarrow$ n^2 even/odd

$n = 2k + i \quad i \in \{0,1\} \quad k \in \mathbb{N}$

$$n^2 = (2k+i)^2 = 4k^2 + 4ki + i^2 = 2\underbrace{(2k^2 + 2ki)}_{=\tilde{k} \in \mathbb{N}} + i^2$$

$$= 2\tilde{k} + i^2 = 2\tilde{k} + i$$

There is no largest prime p^{max} $\rightarrow$ set of all primes $\mathbb{P}$

Lemma: n divisible by $k^{\neq 1}$ $\Rightarrow$ $n-1$ is not divisible by k

Assume there is a largest prime p^{max}

$\prod \mathbb{P} = p^*$ $\rightarrow$ $p^* - 1$ is not divisible by $\underline{\text{any}}$ $p \in \mathbb{P}$

1) is not divisible by any $x \in \mathbb{N}$

2) $\rightarrow$ $p^* - 1$ prime $\wedge$ $p^* - 1 > p^{max}$ $\perp$

$\exists$ $\underline{k}$, $k \in \mathbb{N}$ divides $p^* - 1$

at least one

$\rightarrow$ k prime $\wedge$ $k \notin \mathbb{P}$ $\perp$

$\Rightarrow$ $\exists$ largest prime is untrue

$\Rightarrow$ $\nexists$ largest prime is true

(Complete) Induction $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

1) Induction ~~base~~ $n=1$: $\sum_{1}^{1} = 1 = \frac{1 \cdot 2}{2} = 1$

$n=2$: $\sum_{1}^{2} = 3 = \frac{2 \cdot 3}{2} = 3$

2) Induction step Assume it's true for n

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n+1 = \frac{n(n+1)}{2} + n+1 = \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{(n+2)(n+1)}{2}$$

By complete induction $\sum^{n} i = \frac{n(n+1)}{2}$

That’s all Folks!

Please take a look at the course material and review any fundamentals you feel unsure about.

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