# Chapter 1: Linear Algebra I

## Vector Spaces

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# Outline

In this chapter, we discuss

- The basic concept of the general, formal vector space concept
- The details of the widely used Euclidean Vector Space
- Mathematical distance functions and their properties
- Limits and continuity beyond univariate real-valued functions
- Key properties of general sets (open/closed, bounded, convex)

# What is a Vector Space?

## General Vector Space

Collection of objects on which we can perform linear operations (addition & scaling)

- Set of vectors $V$ and scalars $F$ with definitions for $v + \tilde{v} \in V$ and $f \cdot v \in V$
- Addition is associative and commutative, scaled addition is distributive
- Additive identity ($v + \mathbf{0} = v$) and inverse ($v + -v = \mathbf{0}$), scaling identity ($\mathbf{1} \cdot v = v$)

$\Rightarrow$ general notation $\mathbb{X} = (X, F, +, \cdot)$ [simplify to $\mathbb{X} = (X, +, \cdot)$ for $F = \mathbb{R}$ ]

## Euclidean Vector Space

Resctrict attention to $\mathbb{R}^n$ with ordered $n-$tuple of objects, e.g. $(2, 1, 3)'(\neq (3, 2, 2)')$

- We destinguish row vs. column vector (convention: “vector” = column vector)
- In above language: $F = \mathbb{R}$ and $V = \mathbb{R}^n := \{(x_1, \dots, x_n)' : (\forall i \in \{1, \dots, n\} : x_i \in \mathbb{R})\}$

# Linear Combination and Basis

**Linear combination** (LC) of vectors

- z is a LC of $n$ vectors $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in X$:
  $z = \sum_{i=1}^{n} \lambda_i x^{(i)}$ with $\lambda_i \in \mathbb{R} \; \forall i \in \{1, \ldots, n\}$

**Span** = set of linear combinations of $\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$, e.g.:

$$Span\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}\right) = \left\{\begin{pmatrix}\lambda+\mu\\\mu\\0\end{pmatrix} : \lambda, \mu \in \mathbb{R}\right\} = \left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \mathbb{R}^3 : x_3 = 0\right\}$$

**Basis** of a vector space: smallest set of vectors to *span* the space

- Basis is not unique: e.g. $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ vs. $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ for $\mathbb{R}^2$
- Canonical basis $\{e^1, \ldots, e^n\}$ with $e^i_i = 1$ and $e^i_j = 0$
- Dimension of vector space: number of elements in the basis

# Linear Dependence and Linear Independence

- Linear dependence: $S \subseteq X$ set, $x \in X$ vector
  $x$ is linearly dependent of $S$ if it is a LC of elements in $S$, that is, $x \in Span(S)$
- Linear independence (LI): $x \notin Span(S)$ ($x$ is no LC of $S$)
  LI set $B \subseteq X$: every element is LI of the remaining set: $\forall b \in B : (b \notin Span(B \backslash \{b\}))$
  e.g. $\{e_1, e_2\}$: $Span(\{e_1\}) = \{(x, 0)' : x \in \mathbb{R}\} \not\ni e_2$ and vice versa

## Testing Linear Independence

The set of vectors $B = \{b_1, b_2, ..., b_k\}$ are linearly independent if

$$\sum_{j=1}^{k} \lambda_j b_j = \mathbf{0} \Rightarrow (\forall j \in \{1, ..., k\} : \lambda_j = 0).$$

# Example: Linear Independence of the Canonical Basis

The vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$$

are linearly independent, because if $\lambda_1 e_1 + \ldots + \lambda_n e_n = 0$, i.e.

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots + \lambda_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The last vector equation implies that $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$.

# Our First Vector-Based Operation: The Scalar Product

Function $"\cdot" : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}, (x, y) \mapsto x \cdot y = \sum_{i=1}^n x_i y_i$

- Reduces scalars to a scalar (one-dimensional) number
- Also called dot product, also notated $\langle x, y \rangle$ or $x'y$
- $x \cdot x = \sum_{i=1}^n x_i^2$ (“sum of squares”)

- Practical usecase: checking orthogonality

  ...of vectors $x, y : x \cdot y = 0$

  ...of lines $f, g$ in $\mathbb{R}^n$: $(x_f^1 - x_f^2) \cdot (x_g^1 - x_g^2) = 0$

  where $x_f^1, x_f^2$ are *distinct* points on $f$ and $x_g^1, x_g^2$ are *distinct* points on $g$

$\Rightarrow$ underlying reason: For $x, y$ with angle $\theta$ it holds: $x \cdot y \propto \cos\theta$

# Norms – an Introduction to Distances

- Measure distance between to objects

  $\Rightarrow$ e.g. two points, two vectors, vector from origin etc.
- Universal (intuitive) criteria
  - Non-negative, and zero only if we don't have to move
  - Symmetric: same distance from A to B and from B to A
  - Detours increase the distance

## Metric

For $\mathbb{X} = (X, +, \cdot)$, a function $d : X \times X \mapsto \mathbb{R}$ defines a **metric** on $X$ if

1. non-negativity: $\forall x, y \in X : d(x, y) \geq 0$, and $d(x, y) = 0 \Leftrightarrow x = y$
2. symmetry: $\forall x, y \in X : d(x, y) = d(y, x)$
3. triangle inequality: $\forall x, y, z \in X : d(x, y) \leq d(x, z) + d(z, y)$

# Norms – Normed Vector Spaces

- Interested in particular distances: lengths of vectors

## Norm

For $\mathbb{X} = (X, +, \cdot)$, a function $|| \cdot || : X \mapsto \mathbb{R}$ defines a **norm** on $X$ if

1. non-negativity: $\forall x \in X : ||x|| \geq 0$, and $||x|| = 0 \Leftrightarrow x = \mathbf{0}$
2. absolute homogeneity: $\forall x \in X, \lambda \in \mathbb{R} : ||\lambda \cdot x|| = |\lambda| \cdot ||x||$
3. triangle inequality: $\forall x, y \in X : ||x + y|| \leq ||x|| + ||y||$

Then, $(X, || \cdot ||)$ forms a **normed vector space**.

- Examples: p-Norm (in $\mathbb{R}^n$): $||x||_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$

  Max. norm (in $\mathbb{R}^n$: $||x||_\infty = \max\limits_{i \in \{1 \dots n\}} |x_i|$

  Natural Norm (in $\mathbb{R}$): $||x|| = |x|$

# Norms – Norm Induced Metrics

- **Norm-induced metric**: derive metric $d_N$ from norm $\| \cdot \|_N \rightarrow d_N(x, y) := \|x - y\|_N$

  $\Rightarrow$ absolute homogeneity: $\forall x, y \in X \, \forall \lambda \in \mathbb{R} \; d_N(\lambda x, \lambda y) = |\lambda| d_N(x, y)$

  $\Rightarrow$ translation invariance: $\forall x, y, z \in X \; d_N(x + z, y + z) = d_N(x, y)$

- Continuity of the norm implies continuity of norm-induced metrics

  $\Rightarrow$ Continuity of the norm function $\| \cdot \| : X \mapsto \mathbb{R}, x \mapsto \|x\|$:

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 : (\|x - y\| < \delta \Rightarrow \Big| \|x\| - \|y\| \Big| < \varepsilon)$$

  $\Rightarrow$ Follows from inverse triangle inequality $\Big| \|x\| - \|y\| \Big| \leq \|x - y\|$

  $\Rightarrow$ Intuition: points that are closer "$\|x - y\|$" have similar magnitude "$\Big| \|x\| - \|y\| \Big|$"

  $\Rightarrow$ Sequence-continuity: we can pull in limits: $\lim_{n\rightarrow\infty} \|x_n\| = \| \lim_{n\rightarrow\infty} x_n \|$

# Norms – Euclidian Space

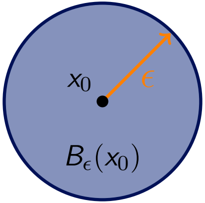
- Focus on Euclidean Vector Spaces $\mathbb{R}^n$ typically considered in Economics
  - Euclidean Norm (p-Norm for p $=$ 2) as standard norm: $\|x\|_2 = \sqrt{x_1^2 + \ldots + x_n^2}$

    $\Rightarrow$ norm-induced metric space $(\mathbb{R}^n, d_2)$ with

$$d_2(x, y) = \|x - y\|_2 = \left( \sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2} = \sqrt{(x-y) \cdot (x-y)}$$

- Crucial importance in Econometrics: Least Squares estimators
- Geometric intuition: direct distance (cf. Pythagorean Theorem)
- “The distance” usually refers to the Euclidean norm(-induced metric)

# Sets and Balls using Metrics



- Ball of radius $\epsilon > 0$ around $x_0$:
  all points with distance to $x_0$ “smaller” than $\epsilon$
  - strictly ($d(x, x_0) < \epsilon$): open ball $B_\epsilon(x_0)$
  - weakly ($d(x, x_0) \leq \epsilon$): closed ball $\bar{B}_\epsilon(x_0)$
  - “closed balls include the boundary, open balls do not”

- Two types of points: interior and boundary points ($int(A)$ vs. $\partial A$)
- Open set: only interior, no boundary points: $A = int(A)$
- Closed set: also includes all boundary points: $A = int(A) \cup \partial A$
- Bounded set: bounded distance of elements: $\exists x \in X \ \exists r < \infty : A \subseteq B_r(x)$
- Compact set $\iff$ closed and bounded [Heine-Borel Theorem]

  $\Rightarrow$ fundamentally important for optimization (together with continuity)

# Sets and Balls using Metrics – Formal Definitions

- $\varepsilon$-open Ball around $x_0$:

$$B_\epsilon(x_0) = \{x \in X : d(x, x_0) < \epsilon\} \quad \text{[Generally]}$$

$$B_\epsilon(x_0) = \{x \in X : \|x - x_0\| < \epsilon\} \quad \text{[Norm-induced metric]}$$

- Interior point:

$$x \in int(A) \Leftrightarrow \exists \epsilon > 0 : B_\epsilon(x) \subseteq A$$

- Boundary point:

$$x \in \partial A \Leftrightarrow \forall \epsilon > 0 : B_\epsilon(x) \cap A \neq \varnothing \wedge B_\epsilon(x) \backslash A \neq \varnothing$$

# Helpful Theorems 1/3

## Properties of Open and Closed Sets

Consider a metric space $(\mathbb{X}, d)$. Then,

(o.i) $\varnothing$ and $X$ are open in $\mathbb{X}$.
(o.ii) A set $A \subseteq X$ is open if and only if its complement $A^c = X \backslash A$ is closed.
(o.iii) The union of an arbitrary (possibly infinite) collection of open sets is open.
(o.iv) The intersection of a finite collection of open sets is open.

(c.i) $\varnothing$ and $X$ are closed in $\mathbb{X}$.
(c.ii) A set $A \subseteq X$ is closed if and only if its complement $A^c = X \backslash A$ is open.
(c.iii) The union of a finite collection of closed sets is closed.
(c.iv) The intersection of a (possibly infinite) collection of closed sets is closed.

Take-away: check complements and/or decompose into $\cup/\cap$ of simple sets!

# Helpful Theorems 2/3

## Closedness and Sequences

Suppose that $\mathbb{X} = (X, +, \cdot)$ is a real vector space, and let $B \subseteq X$. Then, $B$ is closed if and only if, for any convergent sequence $\{x_n\}_{n \in \mathbb{N}}$ over $B$, i.e. $\forall n \in \mathbb{N} : x_n \in B$, it holds that $\lim_{n \to \infty} x_n \in B$.

## Checking Boundedness

Suppose $(\mathbb{X}, d)$ is a metric space where $d$ is norm-induced, i.e. for $x, y \in X$, $d(x, y) = \|x - y\|$. Let $A \subseteq X$. Then, $A$ is bounded if the norm is bounded on $A$, i.e. $\exists b < \infty : (\forall x \in A : \|x\| < b)$.

# Helpful Theorems 3/3

## Weak Inequalities and the Limit: Functions

Suppose that $\mathbb{X} = (X, +, \cdot)$ is a real vector space, $f : X \mapsto \mathbb{R}$ and $g : X \mapsto \mathbb{R}$ so that $\forall x \in X$: $f(x) \leq g(x)$ (in function notation: $f \leq g$). Let $x_0 \in X$, and suppose that $\exists f_0, g_0 \in \mathbb{R}$ so that $\lim_{x \to x_0} f(x) = f_0$, $\lim_{x \to x_0} g(x) = g_0$. Then, it holds that $f_0 \leq g_0$.

## Weak Inequalities and the Limit: Sequences

Suppose that $\mathbb{X} = (X, +, \cdot)$ is a real vector space. Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be convergent sequences over $X$, i.e. $\forall n \in \mathbb{N} : x_n, y_n \in B$, with limits $x \in X$ and $y \in X$, respectively. If $\forall n \in \mathbb{N}$, it holds that $x_n \leq y_n$, then, we also have $x \leq y$.

# Convergence in Normed Spaces – Sequences

- Recall $\mathbb{R}$: real sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent with limit $x \in \mathbb{R}$ if

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : |x_n - x| < \epsilon)$$

- Recall: $|\cdot|$ is the natural norm of the $\mathbb{R}$, so that equivalently

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : \|x_n - x\| < \epsilon)$$

- Extends to **normed VS** $(\mathbb{X}, \|\cdot\|_X)$: $\{x_n\}_{n\in\mathbb{N}}$ where $\forall n \in \mathbb{N} : x_n \in X$ is convergent with limit $x \in X$ if

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : \|x_n - x\|_X < \epsilon)$$

- For sequences over $\mathbb{R}^n$:

$$\lim_{n\to\infty} (x_1, \dots, x_n) = (\lim_{n\to\infty} x_1, \dots, \lim_{n\to\infty} x_n)$$

# Convergence in Normed Spaces – Functions

- Recall: for a univariate, real-valued function, i.e. $f: X \mapsto Y$ with $X, Y \subseteq \mathbb{R}$, $f_a \in Y$ is the limit of $f$ at $a \in X$ if

$$\forall \epsilon > 0 \, \exists \delta > 0 : (\forall x \in X : |x - a| < \delta \Rightarrow |f(x) - f_a| < \epsilon)$$

- General function $f: X \mapsto Y$ where $X \subseteq (\mathbb{X}, \|\cdot\|_X)$, $Y \subseteq (\mathbb{Y}, \|\cdot\|_Y)$:

$$\forall \epsilon > 0 \, \exists \delta > 0 : (\forall x \in X : \|x - a\|_X < \delta \Rightarrow \|f(x) - f_a\|_Y < \epsilon)$$

$\Rightarrow$ Can equivalently write $x \in B_\delta(a)$ for $\|x - a\|_X < \delta$

- More general definitions for any metric space (not “just” norm-induced) exist, less relevant to us

# Convergence in Normed Spaces – Continuity

- Recall: continuity for a univariate, real-valued function $f(a) = \lim_{x \to a} f(x)$

$$\forall \epsilon > 0 \, \exists \delta > 0 : (\forall x \in B_\delta(x_0) : |f(x) - f(x_0)| < \epsilon)$$

- General function $f : X \mapsto Y$ where $X \subseteq (\mathbb{X}, \| \cdot \|_X)$, $Y \subseteq (\mathbb{Y}, \| \cdot \|_Y)$:

$$\forall \epsilon > 0 \, \exists \delta > 0 : (\forall x \in B_\delta(x_0) : \|f(x) - f(x_0)\|_Y < \epsilon)$$

- For continuous functions limits can be pulled in:

$$\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x) = f(x_0)$$

- Disprove continuity:

  find $x_n \overset{n \to \infty}{\to} x_0$ with $f(x_n) \overset{n \to \infty}{\not\to} f(x_0)$ (non-existent or different limit)

# Convexity

- Often domains aren't VS $\Rightarrow$ use sets that focus on special linear combinations

## Convex Combination, Convex Set

Suppose $\mathbb{X}$ is a real vector space. A convex combination $x^c$ of the vectors $x_1, \ldots, x_n \in X$ is a LC $x^c = \sum_{i=1}^n \lambda_i x_i$, for which $\forall i \in \{1, \ldots, n\} : \lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$.
A set $A \subseteq X$ is convex if it contains all convex combinations of any two of its elements, i.e. $\forall a_1, a_2 \in A \;\; \forall \lambda \in [0,1] : \lambda a_1 + (1-\lambda) a_2 \in A$.

- Intuition: convex combination of 2 vectors: $\{\lambda x + (1-\lambda) y : \lambda \in [0,1]\}$

  $\Rightarrow$ line connecting $x$ and $y$ $\rightarrow$ the larger $\lambda$, the more we move from $y$ to $x$
- Graphical test in $\mathbb{R}^2$ and $\mathbb{R}^3$: connecting lines fully contained in set?

# Convexity-preserving Operations

## Convexity-preserving Operations

Suppose $\mathbb{X} = (X, +, \cdot)$ is a real vector space. Then,

1. $\varnothing$ and $X$ are convex.
2. if $A \subseteq X$ is convex, then so is $\alpha A := \{\alpha \cdot a : a \in A\}$ for any $\alpha \in \mathbb{R}$.
3. if $A, B \subseteq X$ are convex, then so is $A + B := \{a + b : a \in A, b \in B\}$.
4. if $\{A_i\}_{i \in I}$ is a (possibly infinite) collection of convex sets, then $\bigcap_I A_i$ is convex.

- Can be used to simplify analysis of sets

# In-Class Exercises I

**Question 1** If $x = (1, 2, 4)'$ and $y = (3, 0, 2)'$, what is $(2x) \cdot y$ ?

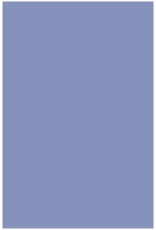
**Question 2** Consider $x = (1, 2, 0)'$ and $y = (2, 4, 1)'$. Are they linearly independent? What is their span? Are additional vectors required to span $\mathbb{R}^3$?

**Question 3** Draw the French metro distance $d_F$ (defined below). Is it a metric?

$$d_F(x, y) = \|x - y\| \quad \text{if } \exists \alpha \in \mathbb{R} : x = \alpha y \quad \text{or} \quad d_F(x, y) = \|x\| + \|y\| \quad \text{otherwise}$$

**Question 4** Which of the following sets are convex?

(A)



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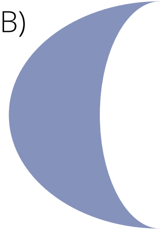

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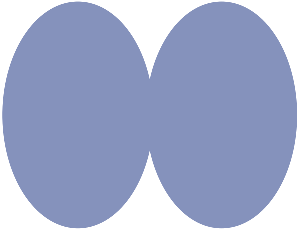

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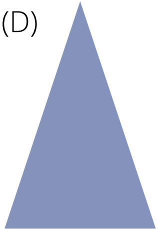

# Recap Chapter 1

**Vector Spaces** Generalization of linear algebra to broader objects

- Linear Combinations and bases form the VS building blocks
- Norms and metrics structure the objects

**Limits and Continuity** Translating defintions to Vector Spaces

- A more general distance concept extends the univariate definitions
- Fundamental concepts remain unchanged

**Set Properties**

- Adjusted definitions for open/closed, compact, convex sets
- Relevance of ”regular” sets for handling and optimization

# That’s all Folks!

**Please take a look at the problem set that we will discuss tomorrow morning.**

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