

Chapter 1: Linear Algebra I

Vector Spaces

Julian Klix | E600 Mathematics

Fall Semester 2026



Outline

In this chapter, we discuss

- The basic concept of the general, formal vector space concept
- The details of the widely used Euclidean Vector Space
- Mathematical distance functions and their properties
- Limits and continuity beyond univariate real-valued functions
- Key properties of general sets (open/closed, bounded, convex)

What is a Vector Space?

General Vector Space

Collection of objects on which we can perform **linear operations** (addition & scaling)

- Set of **vectors** V and **scalars** F with definitions for $v + \tilde{v} \in V$ and $f \cdot v \in V$
- Addition is associative and commutative, scaled addition is distributive
- Additive **identity** ($v + \mathbf{0} = v$) and **inverse** ($v + (-v) = \mathbf{0}$), scaling identity ($1 \cdot v = v$)
 $\Rightarrow$ general notation $\mathbb{X} = (X, F, +, \cdot)$ [simplify to $\mathbb{X} = (X, +, \cdot)$ for $F = \mathbb{R}$]

Euclidean Vector Space

Resctrict attention to $\mathbb{R}^n$ with **ordered** n -tuple of objects, e.g. $(2, 1, 3)'$ ($\neq (3, 1, 2)'$)

- We destinguish row vs. column vector (convention: "vector" = column vector)
- In above language: $F = \mathbb{R}$ and $V = \mathbb{R}^n := \{(x_1, \dots, x_n)' : (\forall i \in \{1, \dots, n\} : x_i \in \mathbb{R})\}$

Vector $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \neq (2 \ 1 \ 3)$

Linear Combination and Basis

Linear combination (LC) of vectors

- z is a LC of k vectors $x^{(1)}, x^{(2)}, \dots, x^{(k)} \in X$:

$$z = \sum_{i=1}^k \lambda_i x^{(i)} \text{ with } \lambda_i \in \mathbb{R} \forall i \in \{1, \dots, k\}$$

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \{2, -1, 5\}$$

$$2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + 5 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Span = set of linear combinations of $\{x^{(1)}, x^{(2)}, \dots, x^{(k)}\}$, e.g.:

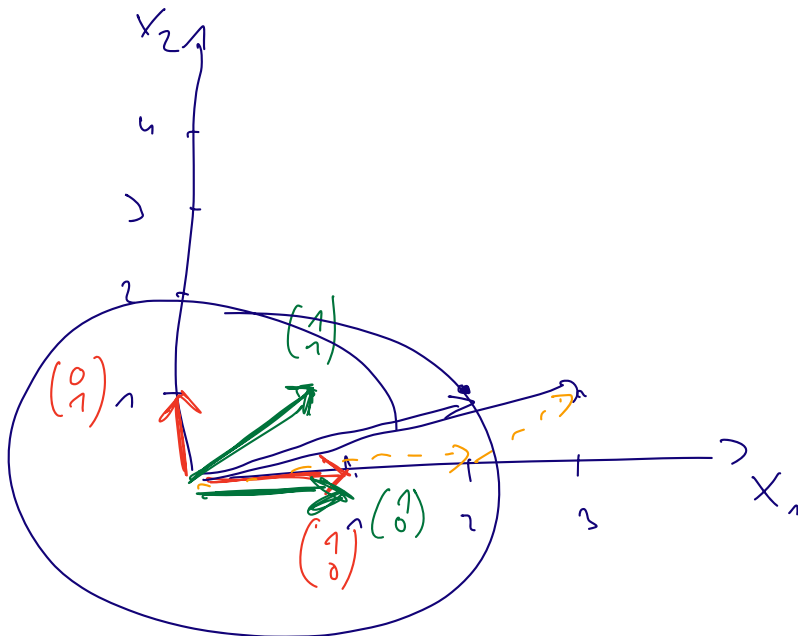
$$\text{Span} \left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \right) = \left\{ \begin{pmatrix} \lambda + \mu \\ \mu \\ 0 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_3 = 0 \right\} = 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Basis of a vector space: smallest set of vectors to *span* the space

- Basis is not unique: e.g. $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ vs. $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ for $\mathbb{R}^2$
- Canonical basis $\{e^1, \dots, e^n\}$ with $e_i^i = 1$ and $e_j^i = 0$
- Dimension of vector space: number of elements in the basis

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

L. ind.

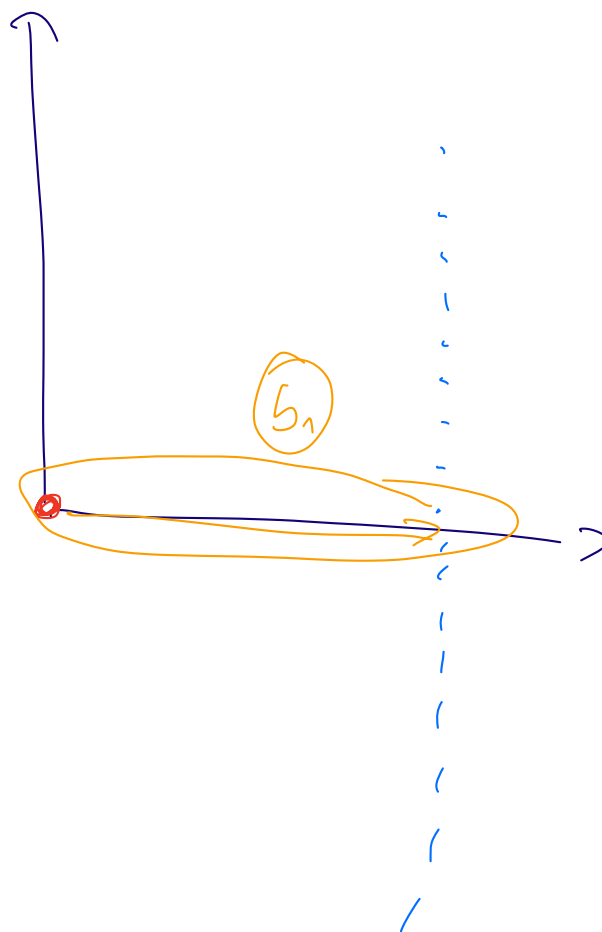


$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



Linear Dependence and Linear Independence

- Linear dependence: $S \subseteq X$ set, $x \in X$ vector
 x is linearly dependent of S if it is a LC of elements in S , that is, $x \in \text{Span}(S)$
- Linear independence (LI): $x \notin \text{Span}(S)$ (x is no LC of S)
 LI set $B \subseteq X$: every element is LI of the remaining set: $\forall b \in B : (b \notin \text{Span}(B \setminus \{b\}))$
 e.g. $\{e_1, e_2\}$: $\text{Span}(\{e_1\}) = \{(x, 0)' : x \in \mathbb{R}\} \not\ni e_2$ and vice versa

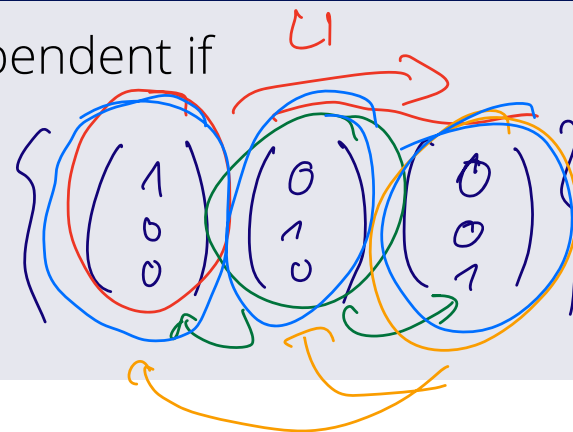
Testing Linear Independence

The set of vectors $B = \{b_1, b_2, \dots, b_k\}$ are linearly independent if

$$\sum_{j=1}^k \lambda_j b_j = \mathbf{0} \Rightarrow (\forall j \in \{1, \dots, k\} : \lambda_j = 0)$$

$$b_k = \lambda_1 b_1 + \lambda_2 b_2 + \dots$$

$$0 = \lambda_1 b_1 + \dots + \lambda_k b_k$$



Example: Linear Independence of the Canonical Basis

The vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n \quad \rightarrow \text{LI set}$$

are linearly independent, because if $\lambda_1 \mathbf{e}_1 + \dots + \lambda_n \mathbf{e}_n = \mathbf{0}$, i.e.

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The last vector equation implies that $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$.

Our First Vector-Based Operation: The Scalar Product

$$f: X \rightarrow Y$$

Function $\cdot: \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}, (x, y) \mapsto x \cdot y = \sum_{i=1}^n x_i y_i$

- Reduces vectors to a scalar (one-dimensional) number
- Also called dot product, also notated $\langle x, y \rangle$ or $x'y$
- $x \cdot x = \sum_{i=1}^n x_i^2$ ("sum of squares")

- Practical usecase: checking orthogonality

...of vectors $x, y: x \cdot y = 0$

...of lines f, g in $\mathbb{R}^n: (x_f^1 - x_f^2) \cdot (x_g^1 - x_g^2) = 0$

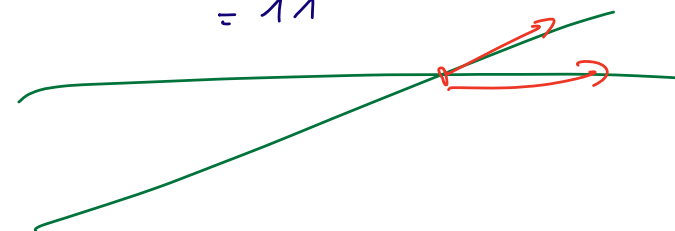
where x_f^1, x_f^2 are *distinct* points on f and x_g^1, x_g^2 are *distinct* points on g

$\Rightarrow$ underlying reason: For x, y with angle θ it holds: $x \cdot y \propto \cos \theta$

$$A \times B = S(a, b)$$

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$= 2 \cdot 3 + 1 \cdot 2 + 3 \cdot 1 \\ = 11$$



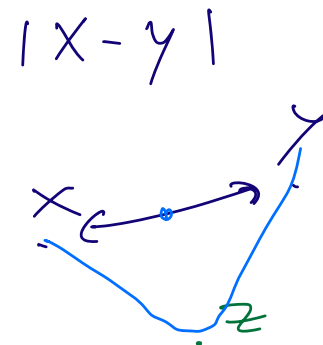
Norms – an Introduction to Distances

- Measure distance between to objects
 $\Rightarrow$ e.g. two points, two vectors, vector from origin etc.

- Universal (intuitive) criteria



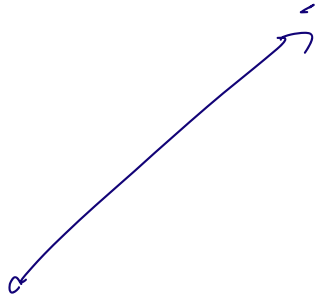
- Non-negative, and zero only if we don't have to move
- Symmetric: same distance from A to B and from B to A
- Detours increase the distance



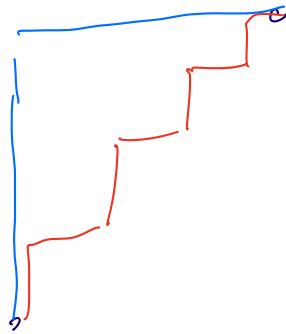
Metric

For $\mathbb{X} = (X, +, \cdot)$, a function $d: X \times X \rightarrow \mathbb{R}$ defines a **metric** on X if

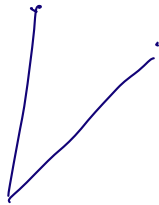
1. non-negativity: $\forall x, y \in X: d(x, y) \geq 0$, and $d(x, y) = 0 \Leftrightarrow x = y$
2. symmetry: $\forall x, y \in X: d(x, y) = d(y, x)$
3. triangle inequality: $\forall x, y, z \in X: d(x, y) \leq d(x, z) + d(z, y)$



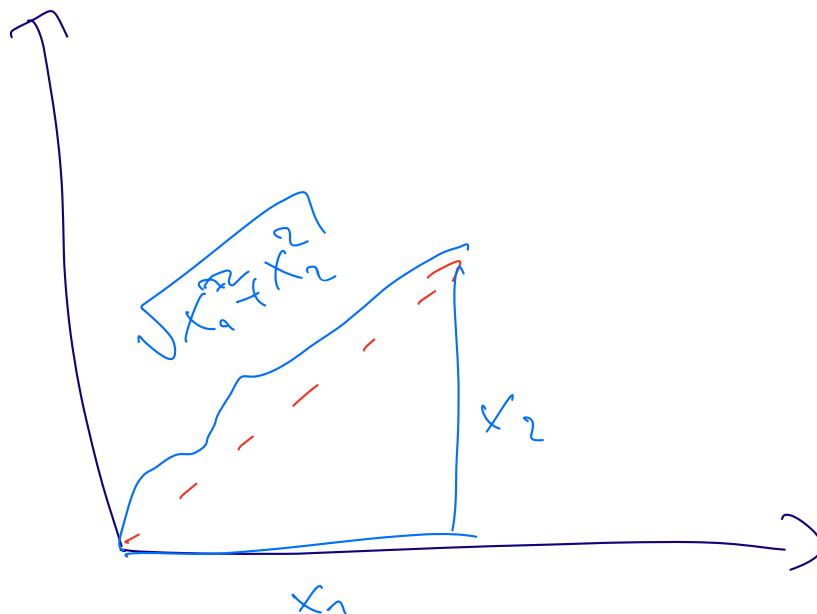
Euclidian
distance



Manhattan (taxi)
distance



Metro distance



Norms – Normed Vector Spaces

- Interested in particular distances: lengths of vectors $\rightarrow$ "distance from zero"

Norm

For $\mathbb{X} = (\mathbf{X}, +, \cdot)$, a function $\|\cdot\| : \mathbf{X} \mapsto \mathbb{R}$ defines a **norm** on $\mathbf{X}$ if

1. non-negativity: $\forall x \in \mathbf{X} : \|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = \mathbf{0}$
2. absolute homogeneity: $\forall x \in \mathbf{X}, \lambda \in \mathbb{R} : \|\lambda \cdot x\| = |\lambda| \cdot \|x\|$
3. triangle inequality: $\forall x, y \in \mathbf{X} : \|x + y\| \leq \|x\| + \|y\|$

homogeneity
of degree 1

Then, $(\mathbf{X}, \|\cdot\|)$ forms a **normed vector space**.

- Examples: **p-Norm** (in $\mathbb{R}^n$): $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$

Max. norm (in $\mathbb{R}^n$): $\|x\|_\infty = \max_{i \in \{1 \dots n\}} |x_i|$

Natural Norm (in $\mathbb{R}$): $\|x\| = |x|$

ℓ^1 -norm

$$\ell^2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots}$$

Norms – Norm Induced Metrics

$$d(x, y) = \|x - y\|$$

- **Norm-induced metric**: derive metric d_N from norm $\|\cdot\|_N \rightarrow d_N(x, y) := \|x - y\|_N$

$\Rightarrow$ absolute homogeneity: $\forall x, y \in X \forall \lambda \in \mathbb{R} \ d_N(\lambda x, \lambda y) = |\lambda| d_N(x, y)$

$\Rightarrow$ translation invariance: $\forall x, y, z \in X \ d_N(x + z, y + z) = d_N(x, y)$

- Continuity of the norm implies continuity of norm-induced metrics

$\Rightarrow$ Continuity of the norm function $\|\cdot\| : X \mapsto \mathbb{R}, x \mapsto \|x\|$:

$$\forall \varepsilon > 0 \ \exists \delta > 0 : (\|x - y\| < \delta \Rightarrow \left| \|x\| - \|y\| \right| < \varepsilon)$$

$\Rightarrow$ Follows from inverse triangle inequality $\left| \|x\| - \|y\| \right| \leq \|x - y\|$

$\Rightarrow$ Intuition: points that are closer " $\|x - y\|$ " have similar magnitude " $\left| \|x\| - \|y\| \right|$ "

$\Rightarrow$ Sequence-continuity: we can pull in limits: $\lim_{n \rightarrow \infty} \|x_n\| = \left\| \lim_{n \rightarrow \infty} x_n \right\|$

$$\text{cont. f} \quad \lim f(x) = f(\lim x)$$

$$\|x + y\| \leq \|x\| + \|y\|$$

Reverse Triangle inequality

$$|\|x\| - \|y\|| \leq \|x - y\|$$

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$$

$$\|x\| - \|y\| \leq \|x - y\|$$

$$\|y\| - \|x\| \leq \|y - x\|$$

Norms – Euclidian Space

- Focus on Euclidean Vector Spaces $\mathbb{R}^n$ typically considered in Economics

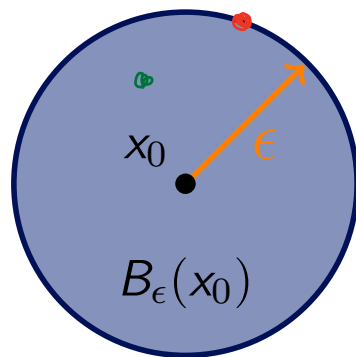
- Euclidean Norm (p-Norm for $p = 2$) as standard norm: $\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$

⇒ norm-induced metric space $(\mathbb{R}^n, d_2)$ with

$$d_2(x, y) = \|x - y\|_2 = \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2} = \sqrt{\underbrace{(x - y) \cdot (x - y)}_{\text{Scalar product}}}$$

- Crucial importance in Econometrics: Least Squares estimators
- Geometric intuition: direct distance (cf. Pythagorean Theorem)
- “The distance” usually refers to the Euclidean norm(-induced metric)

Sets and Balls using Metrics



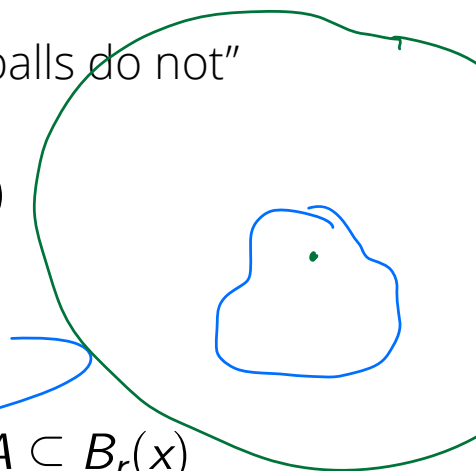
Neighborhood



- Ball of radius $\epsilon > 0$ around x_0 :
all points with distance to x_0 "smaller" than ϵ
 - strictly ($d(x, x_0) < \epsilon$): open ball $B_\epsilon(x_0)$
 - weakly ($d(x, x_0) \leq \epsilon$): closed ball $\bar{B}_\epsilon(x_0)$
 - "closed balls include the boundary, open balls do not"

- Two types of points: interior and boundary points ($\text{int}(A)$ vs. ∂A)
- Open set: only interior, no boundary points: $A = \text{int}(A)$
- Closed set: also includes all boundary points: $A = \text{int}(A) \cup \partial A$
- Bounded set: bounded distance of elements: $\exists x \in X \exists r < \infty : A \subseteq B_r(x)$

- Compact set $\iff$ closed and bounded [Heine-Borel Theorem] $\rightarrow$ Euclidian Spaces
 $\Rightarrow$ fundamentally important for optimization (together with continuity)



Sets and Balls using Metrics – Formal Definitions

- ε -open Ball around x_0 :

$$B_\varepsilon(x_0) = \{x \in X : d(x, x_0) < \varepsilon\} \quad [\text{Generally}]$$

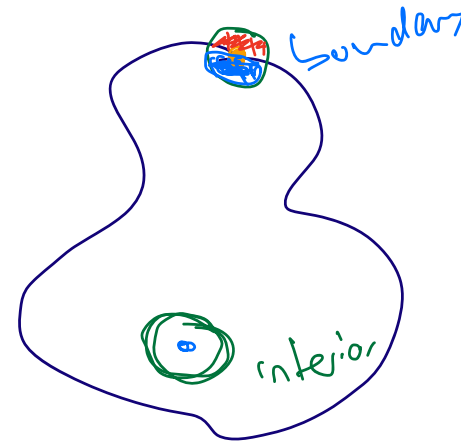
$$B_\varepsilon(x_0) = \{x \in X : \|x - x_0\| < \varepsilon\} \quad [\text{Norm-induced metric}]$$

- Interior point:

$$x \in \text{int}(A) \Leftrightarrow \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A$$

- Boundary point:

$$x \in \partial A \Leftrightarrow \forall \varepsilon > 0 : B_\varepsilon(x) \cap A \neq \emptyset \wedge B_\varepsilon(x) \setminus A \neq \emptyset$$



Helpful Theorems 1/3

Vacuous truth

Properties of Open and Closed Sets

$\rightarrow$ closed

Consider a metric space $(\mathbb{X}, d)$. Then,

(o.i) $\emptyset$ and X are open in $\mathbb{X}$.

(o.ii) A set $A \subseteq X$ is open if and only if its complement $A^c = X \setminus A$ is closed.

(o.iii) The union of an arbitrary (possibly infinite) collection of open sets is open.

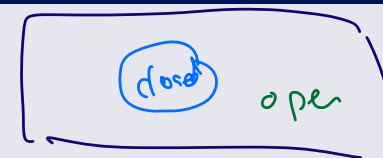
(o.iv) The intersection of a finite collection of open sets is open.

(c.i) $\emptyset$ and X are closed in $\mathbb{X}$.

(c.ii) A set $A \subseteq X$ is closed if and only if its complement $A^c = X \setminus A$ is open.

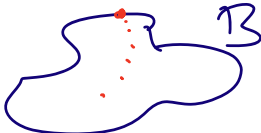
(c.iii) The union of a finite collection of closed sets is closed.

(c.iv) The intersection of a (possibly infinite) collection of closed sets is closed.



Take-away: check complements and/or decompose into $\cup/\cap$ of simple sets!

Helpful Theorems 2/3

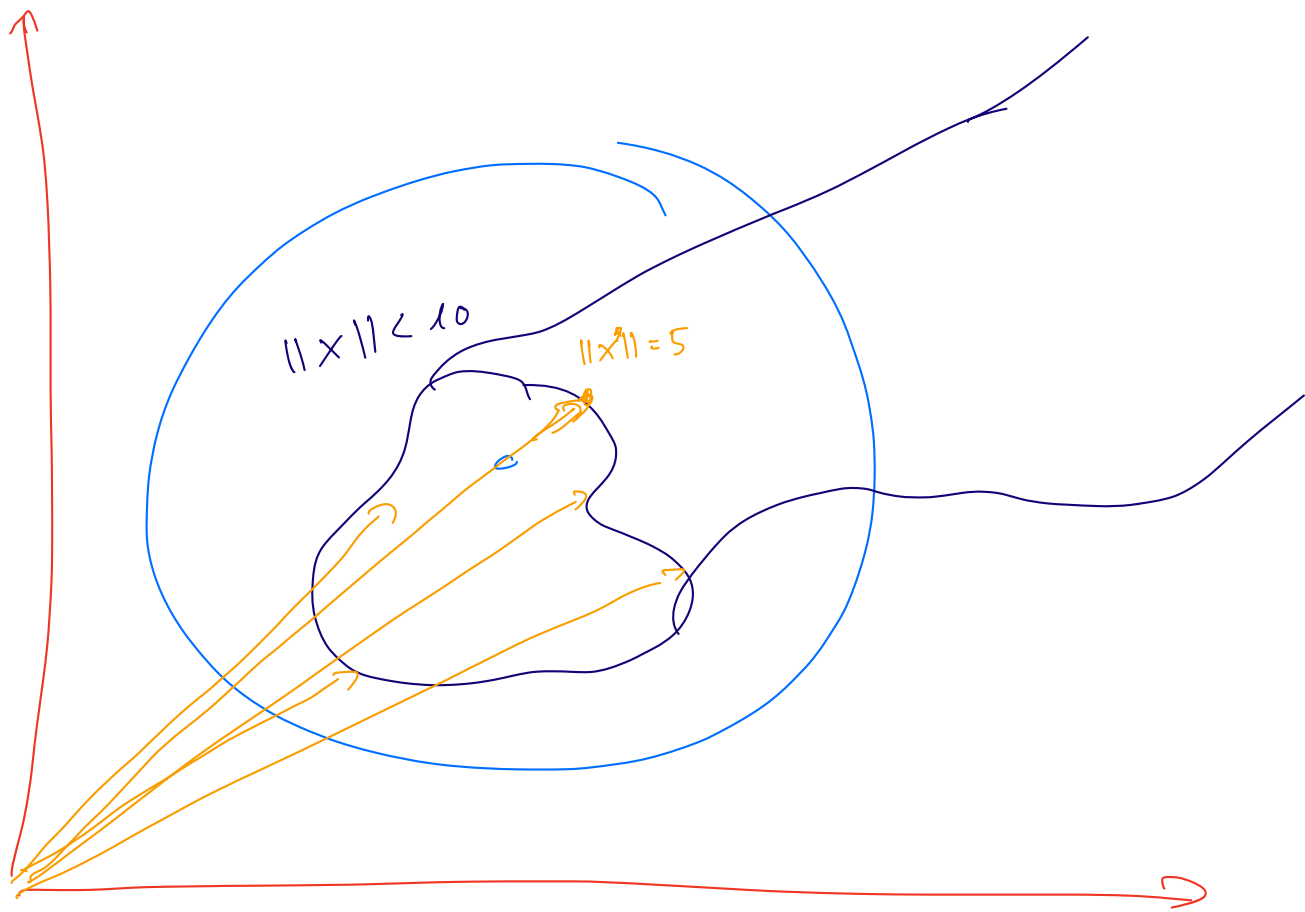
$$(0, \infty) \ni x_n = \frac{1}{n} \rightarrow 0$$


Closedness and Sequences

Suppose that $\mathbb{X} = (X, +, \cdot)$ is a real vector space, and let $B \subseteq X$. Then, B is closed if and only if, for any convergent sequence $\{\underline{x_n}\}_{n \in \mathbb{N}}$ over B , i.e. $\forall n \in \mathbb{N} : \underline{x_n} \in B$, it holds that $\lim_{n \rightarrow \infty} x_n \in B$.

Checking Boundedness

Suppose $(\mathbb{X}, d)$ is a metric space where d is norm-induced, i.e. for $x, y \in X$, $d(x, y) = \|x - y\|$. Let $A \subseteq X$. Then, A is bounded if the norm is bounded on A , i.e. $\exists b < \infty : (\forall x \in A : \|x\| < b)$.



Helpful Theorems 3/3

$$f(x) \leq g(x) \quad x \rightarrow x_0 \quad f(x_0) \leq g(x_0)$$

Weak Inequalities and the Limit: Functions

Suppose that $\mathbb{X} = (X, +, \cdot)$ is a real vector space, $f: X \mapsto \mathbb{R}$ and $g: X \mapsto \mathbb{R}$ so that $\forall x \in X: f(x) \leq g(x)$ (in function notation: $f \leq g$). Let $x_0 \in X$, and suppose that $\exists f_0, g_0 \in \mathbb{R}$ so that $\lim_{x \rightarrow x_0} f(x) = f_0$, $\lim_{x \rightarrow x_0} g(x) = g_0$. Then, it holds that $f_0 \leq g_0$.

Weak Inequalities and the Limit: Sequences

Suppose that $\mathbb{X} = (X, +, \cdot)$ is a real vector space. Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be convergent sequences over X , i.e. $\forall n \in \mathbb{N} : x_n, y_n \in B$, with limits $x \in X$ and $y \in X$, respectively. If $\forall n \in \mathbb{N}$, it holds that $x_n \leq y_n$, then, we also have $x \leq y$.

Sandwiching
& Weier

$$z_n \Rightarrow ? \quad x_n < z_n < y_n \quad \begin{matrix} x_n \rightarrow x \\ y_n \rightarrow y \end{matrix}$$

Convergence in Normed Spaces – Sequences

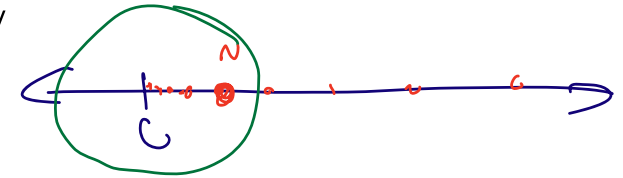
- Recall $\mathbb{R}$: real sequence $\{x_n\}_{n \in \mathbb{N}}$ is convergent with limit $x \in \mathbb{R}$ if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : \|x_n - x\| < \epsilon)$$

in $\mathbb{R}$: $\|x\| = |x|$

- Recall: $|\cdot|$ is the **natural norm** of $\mathbb{R}$, so that equivalently

$$\forall \epsilon > 0 \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : \|x_n - x\| < \epsilon)$$



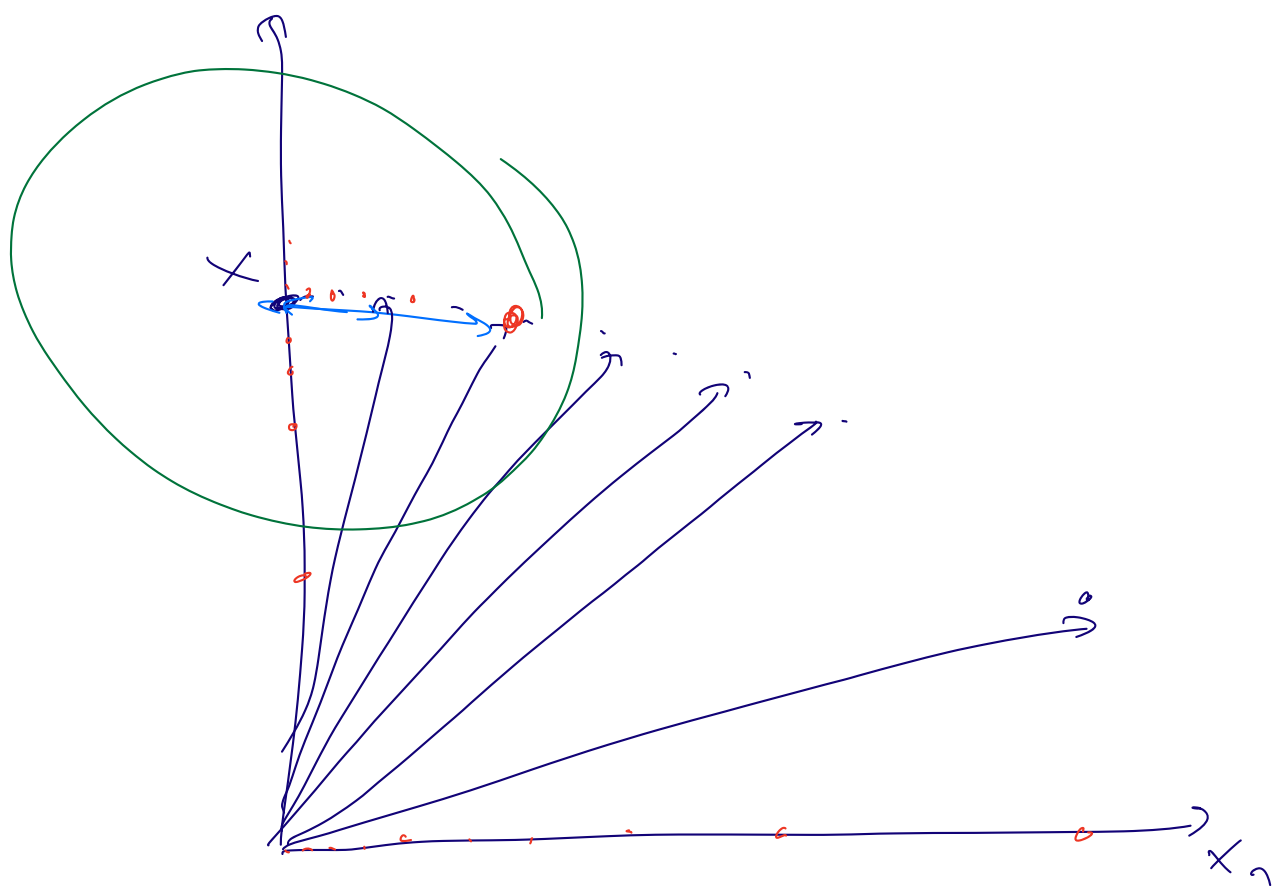
- Extends to **normed VS** $(X, \|\cdot\|_X)$: $\{x_n\}_{n \in \mathbb{N}}$ where $\forall n \in \mathbb{N} : x_n \in X$ is convergent with limit $x \in X$ if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : \|x_n - x\|_X < \epsilon)$$

$$x_n = \left(\frac{1}{n}, \sum_{k=1}^n \left(\frac{1}{k}\right)^n \right) \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x$$

- For sequences over $\mathbb{R}^n$:

$$\lim_{n \rightarrow \infty} (x_1, \dots, x_n) = \left(\lim_{n \rightarrow \infty} x_1, \dots, \lim_{n \rightarrow \infty} x_n \right)$$



Convergence in Normed Spaces – Functions

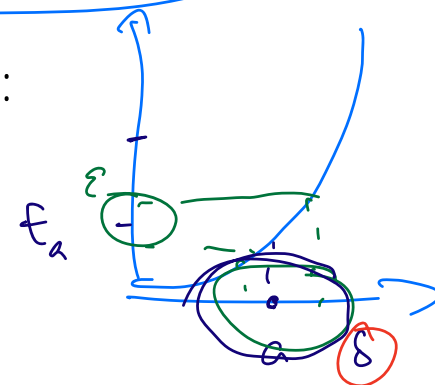
- Recall: for a univariate, real-valued function, i.e. $f: X \mapsto Y$ with $X, Y \subseteq \mathbb{R}$, $f_a \in Y$ is the limit of f at $a \in X$ if

$$\forall \epsilon > 0 \exists \delta > 0 : (\forall x \in X : |x - a| < \delta \Rightarrow |f(x) - f_a| < \epsilon)$$

- General function $f: X \mapsto Y$ where $X \subseteq (\mathbb{X}, \|\cdot\|_X)$, $Y \subseteq (\mathbb{Y}, \|\cdot\|_Y)$:

$$\forall \epsilon > 0 \exists \delta > 0 : (\forall x \in X : \|x - a\|_X < \delta \Rightarrow \|f(x) - f_a\|_Y < \epsilon)$$

$\Rightarrow$ Can equivalently write $x \in B_\delta(a)$ for $\|x - a\|_X < \delta$



- More general definitions for any metric space (not “just” norm-induced) exist, less relevant to us

Convergence in Normed Spaces – Continuity

- Recall: continuity for a univariate, real-valued function $f(a) = \lim_{x \rightarrow a} f(x)$

$$\forall \epsilon > 0 \exists \delta > 0 : (\forall x \in B_\delta(x_0)) : \|f(x) - f(x_0)\| < \epsilon$$

- General function $f: X \mapsto Y$ where $X \subseteq (\mathbb{X}, \|\cdot\|_X)$, $Y \subseteq (\mathbb{Y}, \|\cdot\|_Y)$:

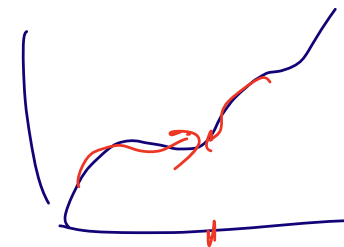
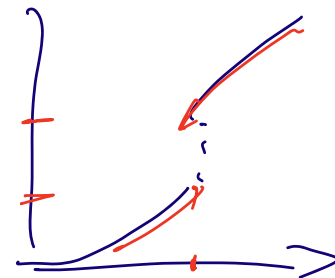
$$\forall \epsilon > 0 \exists \delta > 0 : (\forall x \in B_\delta(x_0)) : \|f(x) - f(x_0)\|_Y < \epsilon$$

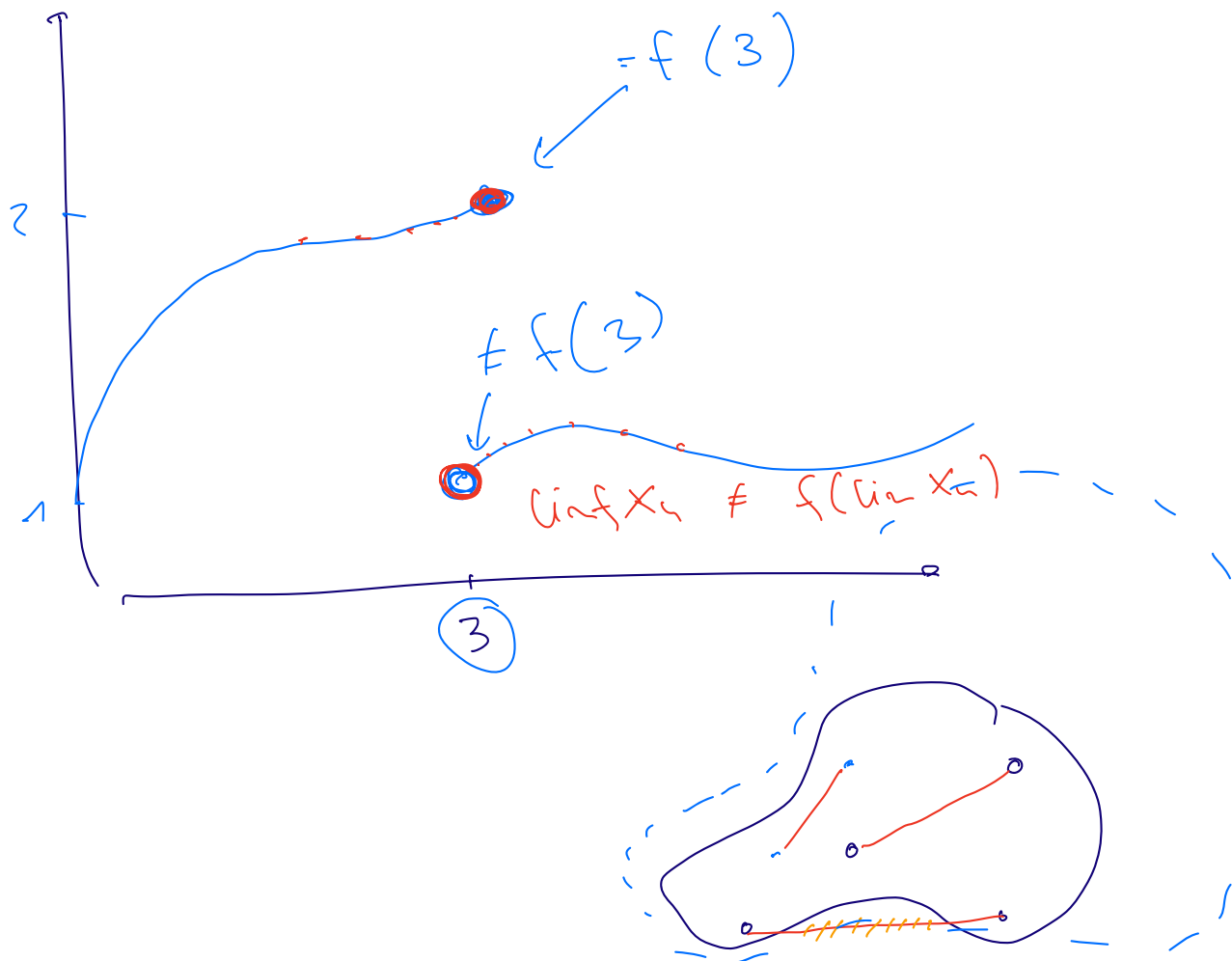
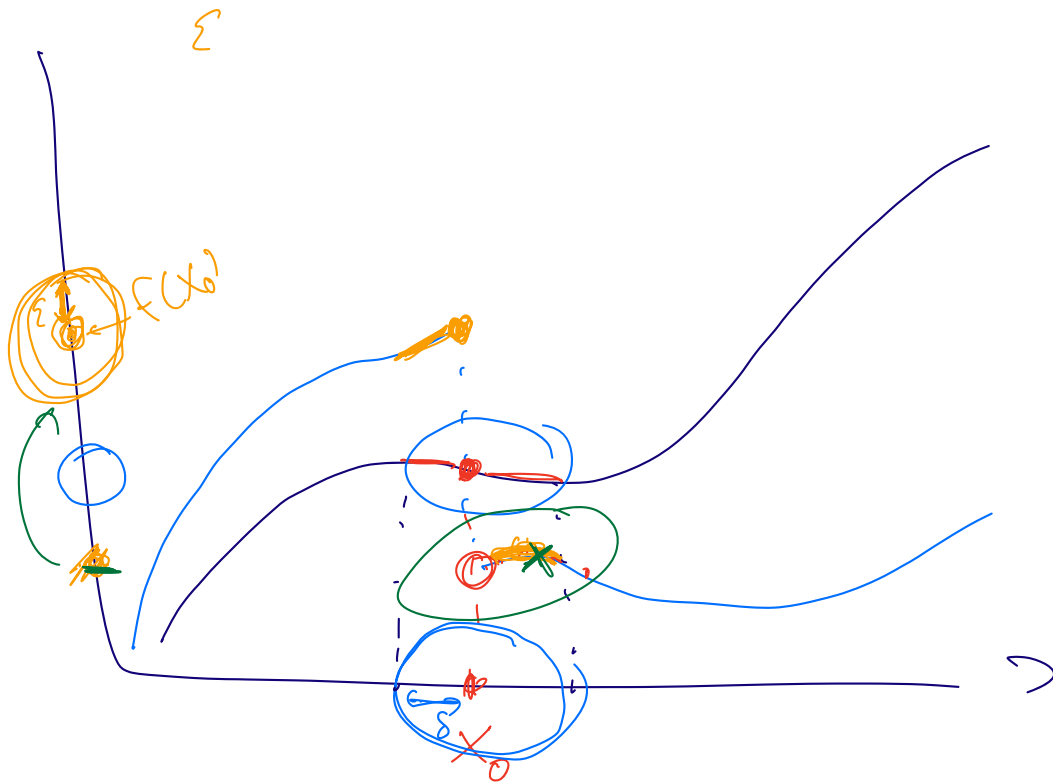
- For continuous functions limits can be pulled in:

$$\lim_{x \rightarrow x_0} f(x) = f(\lim_{x \rightarrow x_0} x) = f(x_0)$$

- Disprove continuity:

$$\text{find } x_n \xrightarrow{n \rightarrow \infty} x_0 \text{ with } f(x_n) \not\xrightarrow{n \rightarrow \infty} f(x_0) \text{ (non-existent or different limit)}$$





Convexity

- Often domains aren't VS $\Rightarrow$ use sets that focus on special linear combinations

Convex Combination, Convex Set

special linear combination

Suppose $\mathbb{X}$ is a real vector space. A convex combination x^c of the vectors $x_1, \dots, x_k \in X$ is a LC $x^c = \sum_{i=1}^k \lambda_i x_i$, for which $\forall i \in \{1, \dots, k\} : \lambda_i \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$.

A set $A \subseteq X$ is convex if it contains all convex combinations of any two of its elements, i.e. $\forall a_1, a_2 \in A \quad \forall \lambda \in [0, 1] : \lambda a_1 + (1 - \lambda) a_2 \in A$.

- Intuition: convex combination of 2 vectors: $\{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$
 $\Rightarrow$ line connecting x and $y \rightarrow$ the larger λ , the more we move from y to x
- Graphical test in $\mathbb{R}^2$ and $\mathbb{R}^3$: connecting lines fully contained in set?

Convexity-preserving Operations

Convexity-preserving Operations

Suppose $\mathbb{X} = (X, +, \cdot)$ is a real vector space. Then,

1. $\emptyset$ and X are convex.
2. if $A \subseteq X$ is convex, then so is $\alpha A := \{\alpha \cdot a : a \in A\}$ for any $\alpha \in \mathbb{R}$.
3. if $A, B \subseteq X$ are convex, then so is $A + B := \{a + b : a \in A, b \in B\}$.
4. if $\{A_i\}_{i \in I}$ is a (possibly infinite) collection of convex sets, then $\bigcap_i A_i$ is convex.

- Can be used to simplify analysis of sets

$$A \cup B$$

$$\underbrace{\{1, 2\}}_A \cup \underbrace{\{10, 20\}}_B = \{1, 2, 10, 20\}$$

$$A + B = \{11, 21, 12, 22\}$$

In-Class Exercises I

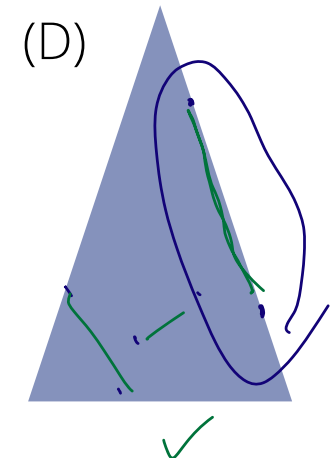
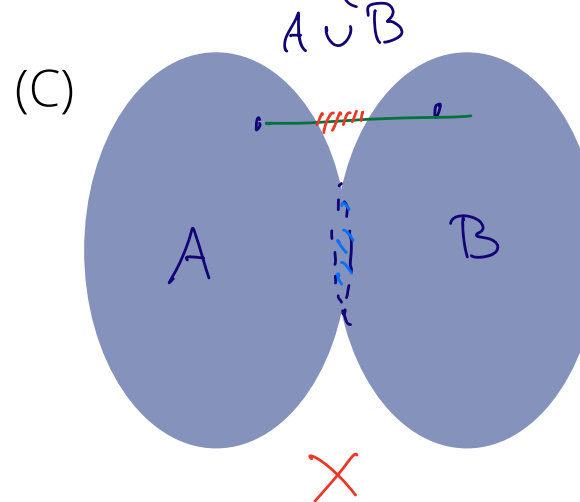
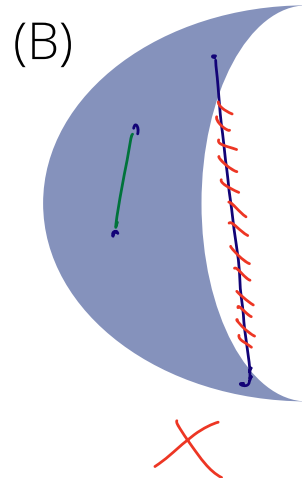
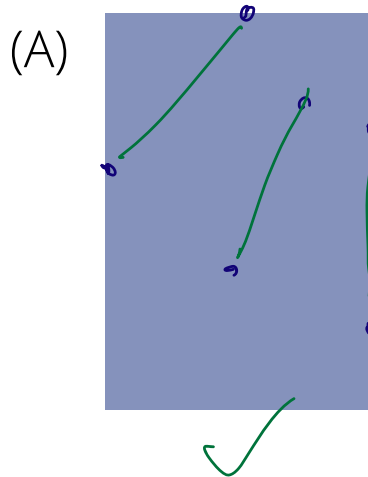
Question 1 If $x = (1, 2, 4)^T$ and $y = (3, 0, 2)^T$, what is $(2x) \cdot y$?

Question 2 Consider $x = (1, 2, 0)^T$ and $y = (2, 4, 1)^T$. Are they linearly independent? What is their span? Are additional vectors required to span $\mathbb{R}^3$?

Question 3 Draw the French metro distance d_F (defined below). Is it a metric?

$$d_F(x, y) = \|x - y\| \text{ if } \exists \alpha \in \mathbb{R} : x = \alpha y \text{ or } d_F(x, y) = \|x\| + \|y\| \text{ otherwise}$$

Question 4 Which of the following sets are convex? (contain their boundary)



$$\textcircled{Q1} \quad x = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \quad y = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \quad \underline{(2x) \cdot y}$$

$$2x = 2 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} \quad y = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = 2 \cdot 3 + 4 \cdot 0 + 8 \cdot 2 = 22$$

$$x = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad y = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 0$$

$$\lambda_1 + 2\lambda_2 = \lambda_1 + 0 = 0$$

$$\Rightarrow \lambda_1 = 0$$

$$\rightarrow \lambda_1 = \lambda_2 = 0$$

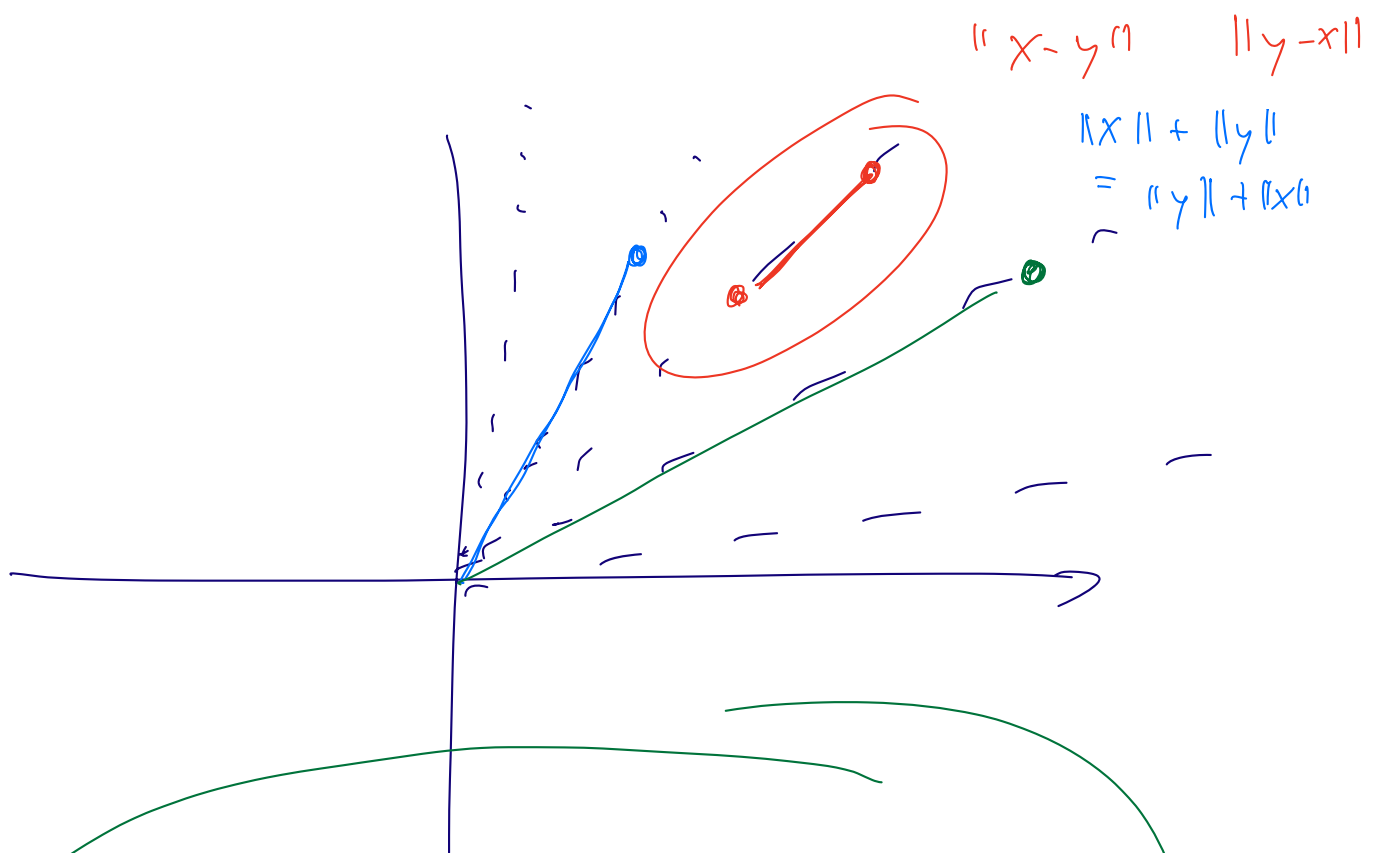
$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ lin. indep.

$$\text{Span} \left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right) \rightarrow \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + 2\lambda_2 \\ 2(\lambda_1 + 2\lambda_2) \\ \lambda_2 \end{pmatrix}$$

$$= \left\{ x \in \mathbb{R}^3 : x = \begin{pmatrix} \alpha \\ 2\alpha \\ \beta \end{pmatrix}, \alpha, \beta \in \mathbb{R} \right\}$$

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



a) non-negativity ✓

b) symmetry ✓

c) triangle inequality ✓

Recap Chapter 1

Vector Spaces Generalization of linear algebra to broader objects

- Linear Combinations and bases form the VS building blocks
- Norms and metrics structure the objects

Limits and Continuity Translating definitions to Vector Spaces

- A more general distance concept extends the univariate definitions
- Fundamental concepts remain unchanged

Set Properties

- Adjusted definitions for open/closed, compact, convex sets
- Relevance of "regular" sets for handling and optimization

Two Group-Chat Links



Official Fachschaft Group



Your Cohort Group

That's all Folks!

Please take a look at the problem set that we will discuss tomorrow morning.

✉ julian.klix@uni-mannheim.de

🏢 L7 3-5, Room 3.43

