# Chapter 2: Linear Algebra II

## Matrix Algebra

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# Outline

In this chapter, we discuss

- The formal matrix concept and key definitions/types of matrices
- The matrix-based linear independence test
- Matrix inversion and its usefulness for solving equation systems
  - understand why matrix inversion works (elementary operations)
  - invert matrices using the Gauss-Jordan algorithm
- Key concepts related to matrices: rank, determinant, eigenvalues, definiteness, and how they are related

# Motivation – Why Matrices?

- Formalize linear operations in Euclidean vector spaces
- Represent systems of linear equations in a simpler (and structured) manner

  $\Rightarrow$ Simplify and regularize solution
- Derivatives of more complex functions "naturally" have matrix shape
- Typical (and practical) way to format empirical data

  $\Rightarrow$ Central object of Econometrics

# Matrices – Formal Definition

## Matrix of dimension n×m

Let $(a_{ij} : i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\})$ be a collection of elements from a set $X$, i.e. $\forall i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\} : a_{ij} \in X$. Then, the matrix $A$ of these elements is the ordered two-dimensional collection

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

We call $n$ the row dimension and $m$ the column dimension of $A$. We write $A \in X^{n \times m}$, and $A = (a_{ij})_{i \in \{1,\ldots,n\}, j \in \{1,\ldots,m\}}$, or, if $n = m$, $A = (a_{ij})_{i,j \in \{1,\ldots,n\}}$.

# Matrices – Representation as a Vector Space

- Real Matrix ($X = \mathbb{R}$) as a vector of vectors:
  - $a_i^r = (a_{i1}, \ldots, a_{im})$ $i$-th row of $A$ (row vector)
  - $a_j^c = (a_{1j}, \ldots, a_{mj})' = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$ $j$-th column of $A$ (column vector)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} a_1^r \\ \vdots \\ a_n^r \end{pmatrix} = (a_1^c, \ldots, a_m^c)$$

  - Typically: leave superscript $r/c$ away if clear from context
- Vector space of real matrices: not vectors of real *numbers* (as $\mathbb{R}^n$), but vectors of real *vectors*!

# Matrices – Basic Algebraic Operations

- Set of real $m \times n$-Matrices denoted as $\mathcal{M}_{n\times m} = \{A \in \mathbb{R}^{n\times m}\}$

## Addition of Matrices

For matrices $A, B$ of **identical dimension**, i.e. $\exists n, m \in \mathbb{N} : A, B \in \mathcal{M}_{n\times m}$, their sum is obtained from addition of their elements, that is

$$A + B = (a_{i,j} + b_{i,j})_{i\in\{1,\dots,n\},j\in\{1,\dots,m\}}$$

## Scalar Multiplication of Matrices

Let $A = (a_{i,j})_{i\in\{1,\dots,n\},j\in\{1,\dots,m\}}$ and $\lambda \in \mathbb{R}$. Then, scalar multiplication of $A$ with $\lambda$ is defined element-wise, that is

$$\lambda A := (\lambda a_{i,j})_{i=1,\dots,n,j=1,\dots,m}$$

# Matrices – Basic Algebraic Operations

## Multiplication of Matrices (Matrix Product)

Consider $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times k}$ where the column dimension of $A$ is equal to the row dimension of $B$. Then, the matrix $C \in \mathbb{R}^{n \times k}$ of column dimension equal to the one of $A$ and row dimension equal to the one of $B$ is called the product of $A$ and $B$, denoted $C = A \cdot B$, where

$$\forall i \in \{1, \ldots, n\}, j \in \{1, \ldots, k\} : c_{ij} = \sum_{l=1}^{m} a_{il} b_{lj}$$

- Matrix Multiplication less straight forward
  - Dimensions of matrices have to match
  - Matrix multiplication is **not** commutative [even for square matrices $A \cdot B \neq B \cdot A$]

# Matrices – Basic Algebraic Operations

$A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 & \cdots & b_k \end{pmatrix}$ (row/column notation). Then,

$$AB = \begin{pmatrix} a_1' \cdot b_1 & a_1' \cdot b_2 & \cdots & a_1' \cdot b_k \\ a_2' \cdot b_1 & a_2' \cdot b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1}' \cdot b_k \\ a_n' \cdot b_1 & \cdots & a_n' \cdot b_{k-1} & a_n' \cdot b_k, \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 4 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 + 3 \cdot 5 & 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 3 \\ 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 5 & 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 3 \\ 4 \cdot 1 + 4 \cdot 0 + 0 \cdot 5 & 4 \cdot 0 + 4 \cdot 1 + 0 \cdot 3 \end{pmatrix} = \begin{pmatrix} 16 & 11 \\ 0 & 1 \\ 4 & 4 \end{pmatrix}$$

# Matrices – Key Properties of the Matrix Product

## Associativity and Distributivity of the Product

Assuming that $A$, $B$, $C$ are matrices of appropriate dimension, their product is

(i) Associative: $(AB)C = A(BC)$ (order of multiplication is irrelevant!)

(ii) Distributive over addition: $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$

## Zero and Identity Matrix

Let $A \in \mathbb{R}^{n \times m}$. Then,

(i) $A + \mathbf{0}_{n \times m} = A$.

(ii) For any $k \in \mathbb{N}$, $A \cdot \mathbf{0}_{m \times k} = \mathbf{0}_{n \times k}$ and $\mathbf{0}_{k \times n} \cdot A = \mathbf{0}_{k \times m}$.

(iii) For any $k \in \mathbb{N}$, $A \cdot \mathbf{I_m} = A$ and $\mathbf{I_n} \cdot A = A$.

# Matrices – Transposition

- **Transposed** matrix $A'/A^T/A^t$: swap of row and column index, e.g.

$$\begin{pmatrix} 1 & 0 & 4 \\ 3 & 1 & 2 \end{pmatrix}' = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 4 & 2 \end{pmatrix}$$

## Transposition, Sum, and Product

(i) Let $A, B \in \mathbb{R}^{n \times m}$. Then, $(A + B)' = A' + B'$

(ii) Let $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times k}$. Then, $(AB)' = B'A'$

(iii) If $A \in \mathbb{R}^{1 \times 1}$, then $A$ is actually a scalar and $A' = A$. (?!)

# Matrices – Special Matrices 1

- $A \in \mathbb{R}^{n\times m}$ is **equal** to the matrix $B$ if (and only if) $B \in \mathbb{R}^{n\times m}$ and $\forall i,j: a_{ij} = b_{ij}$

  e.g. $\mathbf{0}_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}_{2\times 3}$

- **Vector** of length $n$: (column) vector $a \in \mathbb{R}^{n\times 1}$, row vector $a \in \mathbb{R}^{1\times n}$

- **Zero matrix**: $\mathbf{0}_{n\times m} \in \mathbb{R}^{n\times m}$: $\mathbf{0}_{n\times m} = (a_{ij})_{i\in\{1,\dots,n\},j\in\{1,\dots,m\}}$ where $\forall i,j: a_{ij} = 0$

- **Square** matrix: $A \in \mathbb{R}^{n\times m}$ with $n = m$, i.e. $A \in \mathbb{R}^{n\times n}$

  Example: $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

# Matrices – Special Matrices 2

- **Symmetric** matrix: *square* matrix $A \in \mathbb{R}^{n \times n}$ with $a_{ij} = a_{ji}\ \forall i, j$, e.g.

$$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \quad \text{but not} \quad \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

- **Diagonal** matrix: $A = (a_{ij})_{i,j \in \{1,\dots,n\}}$ with $(i \neq j \Rightarrow a_{ij} = 0)$, e.g.

$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$ We write $A = diag\{a_{11}, a_{22}, \dots, a_{nn}\}$

- **Identity** matrix: $\mathbf{I_n} = diag\{1, 1, \dots, 1\}$

# Matrices – Special Matrices 3

## Upper and Lower Triangular Matrix

A square matrix $A = (a_{ij})_{i,j \in \{1,\ldots,n\}} \in \mathbb{R}^{n \times n}$ is said to be **upper triangular** if $(i > j \Rightarrow a_{ij} = 0)$, i.e. when the entry $a_{ij}$ equals zero whenever it lies below the diagonal. Conversely, $A$ is said to be lower triangular if $A'$ is upper triangular, i.e. $(i < j \Rightarrow a_{ij} = 0)$.

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \text{and conversely} \quad A' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & -4 & 0 \\ 4 & 0 & 3 & 2 \end{pmatrix}.$$

# Matrices – Matrix Vector Space

- Linear Operations: matrix addition and scalar multiplication as defined before

  $\Rightarrow$ follows regular rules except for (not required) matrix product commutativity

- Existence of multiplicative identity $(1)$, additive identity $(\mathbf{0}_{n\times m})$ and inverse $(-A)$

- Maximum Norm of $(\mathcal{M}_{n\times m}, +, \cdot)$:

  $\|A\|_\infty = \max\{|a_{ij}| : i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$

  1. $\|A\|_\infty \geq 0$ and $(\|A\|_\infty = 0 \Leftrightarrow \forall i, j : a_{ij} = 0$, i.e. $A = \mathbf{0}_{n\times m})$
  2. Triangle inequality transfers from $|\cdot|$
  3. Absolute homogeneity follows from $|\cdot|$ as well

$\Rightarrow$ Can define norm-induced metrics, convergence, etc. like for $\mathbb{R}^n$!

# Linear Equation Systems – Matrix Representation

- Translate linear equation system into matrix notation:

$$\begin{array}{ccccccc} 1\cdot x_1 & + & 2\cdot x_2 & + & 3\cdot x_3 & = & 2 \\ 0\cdot x_1 & + & 1\cdot x_2 & + & 4\cdot x_3 & = & 2 \\ 5\cdot x_1 & + & 9\cdot x_2 & + & 0\cdot x_3 & = & 3 \end{array} \iff \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{=x} = \underbrace{\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}}_{=b}$$

- Interpretation: search of arguments $x$ that solve $f_A(x) = b$ for $f_A(x) = Ax$

  $\Rightarrow$ how to use $A$ to characterize the solutions and determine them?
- Easiest scenarios: “square system” – as many equations as unknowns
- Simplest of all: one equation in one unknown: $f_a(x) = ax = b$, $a, b \in \mathbb{R}$
  - If $a \neq 0$, then unique solution $x = a^{-1}b = b/a$
  - If $a = 0$ and $b = 0$: infinitely many solutions $x \in \mathbb{R}$
  - If $a = 0$ and $b \neq 0$: no solution (“$b$ never reached by $a \cdot x$”)

# Linear Equation Systems – Solution Idea

- $1 \times 1$ system: generalization step
  - Unique solution if and only if $a$ is invertible [ $\iff a \neq 0$], i.e.

$$\exists a^{-1} \in \mathbb{R} : a^{-1}a = aa^{-1} = 1$$
$$\rightarrow x^* = a^{-1}b$$

  - Otherwise: infinitely many/no solutions
- For $n \times n$ systems ($Ax = b$):
  - Unique solution if and only if $A$ is invertible, i.e.

$$\exists A^{-1} \in \mathbb{R}^{n \times n} : A^{-1}A = AA^{-1} = \mathbf{I_n}$$
$$\rightarrow x^* = a^{-1}b$$

  - Otherwise: infinitely many/no solutions
- Now: when does an inverse $A^{-1}$ exist?

# Matrix Inversion – Definition and Uniqueness

- Inverse matrix of square matrix $A \in \mathbb{R}^{n \times n}$: $A^{-1} \in \mathbb{R}^{n \times n}$ so that

$$A^{-1}A = AA^{-1} = \mathbf{I_n}$$

- The inverse matrix is unique!
  *Proof.* Let $B, C \in \mathbb{R}^{n \times n}$ with $BA = AB = \mathbf{I_n}$ and $CA = AC = \mathbf{I_n}$. Then, it holds that

$$C = C\mathbf{I_n} = C(AB) = \underbrace{(CA)}_{=\mathbf{I_n}} B = \mathbf{I_n}B = B,$$

  i.e. $C = B$

- How to (dis)prove intertability?
- How to find the inverse?

# Matrix Inversion – The Gauß-Jordan Algorithm

- Process to find inverse by converting a (square) matrix into identity matrix

$$\underbrace{\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)}_{A \qquad\qquad I_3} \longrightarrow \underbrace{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right)}_{I_3 \qquad\qquad A^{-1}}$$

- Allowed operations:
  1. Interchange of (two) rows
  2. Scalar multiplication of a row
  3. Adding (scaled) rows to other rows

# Matrix Inversion – Elementary Matrix Operations

## Elementary Matrix Operations

Let $A$ with rows $a_1', \dots, a_n' \in \mathbb{R}^n$ and $\tilde{A}$ with rows $\tilde{a}_1', \dots, \tilde{a}_n' \in \mathbb{R}^n$ be two square matrices. The three elementary matrix operations are mappings $A \to \tilde{A}$ given by

(E1) Interchange of two rows $i, j$: $\tilde{a}_i = a_j$, $\tilde{a}_j = a_i$ and $\tilde{a}_k = a_k \; \forall k \notin \{i, j\}$,

(E2) Scalar multiplication of a row $i$ with $\lambda \neq 0$: $\tilde{a}_i = \lambda a_i$ and $\tilde{a}_j = a_j \; \forall j \neq i$,

(E3) Addition of a scaled row $j$ to row $i$: $\tilde{a}_i = a_i + \lambda a_j$, $\lambda \in \mathbb{R}$, and $\tilde{a}_j = a_j \; \forall j \neq i$.

- Formalizes the operation used in Gauß-Jordan's algorithm
- Not to be confused with basis operations $+$ and $\cdot$

# Matrix Inversion – Foramlizing Gauß-Jordan

- Elementary Operations represented by left-multiplication: $\tilde{A} = EA = E_2 E_1 A$
- Transforming $A$ to $I_n$ (in $k$ steps): $E_k E_{k-1} \dots E_1 A = \mathbf{I_n}$

  $\Rightarrow$ Recall definition of $A^{-1}$ : $A^{-1}A = I_n$ by uniqueness: $A^{-1} = E_k E_{k-1} \dots E_1$

  $\Rightarrow$ Nothing special about $E_i$ (except relatively easy to find)

Example:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & \end{array}\right)$$

# Matrix Inversion – Helpful Theorems

## Invertability: Transpose and Product

Suppose that $A, B \in \mathbb{R}^{n\times n}$ are invertible. Then,

- $A'$ is invertible and $(A')^{-1} = (A^{-1})'$,
- $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$,
- $\forall \lambda \in \mathbb{R}$, $\lambda \neq 0$, $\lambda A$ is invertible and $(\lambda A)^{-1} = 1/\lambda A^{-1}$.

## Inverse of a $2 \times 2$ Matrix

Consider a $2 \times 2$ matrix $A$. Then, as long as $ad \neq bc$, for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\rightarrow\; A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

# Determinant – Definition

- Submatrix $A_{-ij}$ of $A$: Matrix $A$ without row $a_i^r$ and column $a_j^c$

  Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix} \longrightarrow A_{-23} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

## Determinant

Let $A \in \mathbb{R}^{n \times n}$. Then, we define the determinant of $A$, denoted $\det(A)$ or $|A|$, for all $n \in \mathbb{N} : n \geq 2$ as

$$\det(A) := \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{-ij}) \text{ for } i = 1$$

Otherwise, for $n = 1$, i.e., $A = (a_{11})$, it is given by $\det(A) = a_{11}$.

# Determinant – Laplace Expansion Theorem

- **Laplace Expansion Theorem**: It holds that
  - $\det(A) := \sum_j (-1)^{i+j} a_{ij} \det(A_{-ij})$ for arbitrary $i \in \{1, \ldots, n\}$ (expansion by $i$-th row)
  - $\det(A) = \sum_i (-1)^{i+j} a_{ij} \det(A_{-ij})$ for arbitrary $j \in \{1, \ldots, n\}$ (expansion by $j$-th column)

$\Rightarrow$ Look for rows/columns with many zeros!

$\Rightarrow$ Typically, do Laplace expansions until down to $3 \times 3$ submatrices

- Determinant of "small" matrices:

1. If $n = 2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(A) = ad - bc$.
2. If $n = 3$ and $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then $\det(A) = aei + bfg + cdh - (ceg + bdi + afh)$.

# Determinant – Triangular Matrices

- The determinant of triangular matrices is equal to its trace (product of diagonal elements)

$$\det(A) = a_{11} \det \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} = a_{11} a_{22} \det \begin{pmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{43} & \cdots & a_{4n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

$$= \ldots = \Pi_{i=1}^{n} a_{ii} = \mathrm{tr}(A)$$

- $\det(A) \neq 0 \Leftrightarrow (\forall i \in \{1, \ldots, n\} : a_{ii} \neq 0)$
- Triangular $A$ is invertible if and only if $\det(A) \neq 0$
- This holds also for non-triangular $A$

# Determinant – Elementary Operations

## Determinant and Elementary Operations

Let $A \in \mathbb{R}^{n \times n}$ and $\tilde{A}$ the resulting matrix for the respective EO. Then,

1. for (E1) (interchange of two rows), we have $\det(\tilde{A}) = -\det(A)$
2. for (E2) (row multiplication with a scalar $\lambda \neq 0$), $\det(\tilde{A}) = \lambda \det(A)$
3. for (E3) (addition of multiple of row to another row), $\det(\tilde{A}) = \det(A)$

- For any square matrix $A$:
  - We can use EOs to bring $A$ to a triangular matrix $\tilde{A}$
  - We can use EOs to bring $\tilde{A}$ to $\mathbf{I}_n$ if $\det(\tilde{A}) \neq 0$
  - EOs do not affect the property "det $\neq 0$"! Thus, $\det(\tilde{A}) \neq 0$ if and only if $\det(A) \neq 0$

# Determinant – Helpful Properties

## Determinant and Product Rule

Let $A, B \in \mathbb{R}^{n\times n}$ be two square matrices of equal dimension. Then,

- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = 1/\det(A)$
- $\det(\lambda A) = \lambda^n \det(A)$ for all $\lambda \in \mathbb{R}$

- So far: "$\det(A) \neq 0 \Rightarrow A$ invertible" [$\rightarrow$ product rule allows to show converse]

  $\Rightarrow$ **we can invert A if and only if** $\det(\mathbf{A}) \neq \mathbf{0}$!

  $\Rightarrow$ Square systems $Ax = b$: unique solution $x^* = A^{-1}b \Leftrightarrow \det(A) \neq 0$

**A square matrix is invertible if and only if it has non-zero determinant! In a Square System, we (almost) always use the determinant invertability criterion!**

# Matrix Rank – An Alternative Solution Criterion

- Column rank: Number of LI columns of $A$
- Row rank: Number of LI rows of $A$

  $\Rightarrow$ Theorem: Column rank = row rank =: Rank, denoted rk($A$) or rk $A$

  [Corollary: rank bound rk $A \leq \min\{m, n\}$ for $n \times m$ matrix $A$]
- For square matrices ($m = n$), $A$ has full rank if rk($A$) $= n$
- **The rank is unchanged by elementary operations (E1) to (E3)!**

## Rank Criterion for Linear Equation Systems

Let $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. Then, the system $Ax = b$ has a unique solution if and only if $rk(A) = n$.

# Solvability of Linear Equation Systems – Summary

- Summarizing all we've seen so far:

## Invertability, Rank and Determinant of Square Matrices

Consider square matrix $A \in \mathbb{R}^{n\times n}$. Then, the following statements are equivalent:

1. $A$ is invertible,
2. $\det(A) \neq 0$,
3. $rk\ A = n$,
4. for any $b \in \mathbb{R}^n$, $Ax = b$ has a unique solution for $x$,
5. Any triangular matrix $\tilde{A}$ obtained from applying EOs to $A$ has only non-zero diagonal entries.

# Non-Square Equation Systems

- Underidentified system of equations

  $\Rightarrow n < m$, strictly less equations than unknowns
  - Information deficit $\rightarrow$ solution has degrees of freedom (not fully pinned down)
- Overidentified system of equations

  $\Rightarrow n > m$, strictly more equations than unknowns
  - Information contradiction $\rightarrow$ no solution can exist
  - Information redundancy $\rightarrow$ system of equation can be reduced

## Generalized Rank Condition

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then, the system $Ax = b$ has a unique solution if and only if $b \in Co(A) := Span(a_1^c, \ldots, a_n^c)$ and $rk(A) = n$.

Column space $Co(A)$: space spanned by the columns of $A$

# Testing Linear Independence using Matrices

- Recall from yesterday:

## Testing Linear Independence

The set of vectors $B = \{b_1, b_2, ..., b_k\}$ are linearly independent if

$$\sum_{j=1}^{k} \lambda_j b_j = \mathbf{0} \Rightarrow (\forall j \in \{1, ..., k\} : \lambda_j = 0).$$

- $\sum_{j=1}^{k} \lambda_j b_j = \mathbf{0}$ is a linear equation system !
- Matrix form: $B_{mat}\lambda = \mathbf{0}$ with $\lambda = (\lambda_1, \ldots, \lambda_k)'$, $B_{mat} = (b_1, b_2, ..., b_k)$
- We need (i) a unique solution (ii) equal to $\lambda = \mathbf{0}$

# Testing Linear Independence using Matrices

## Testing Linear Independence

The set of vectors $B = \{b_1, b_2, ..., b_k\}$ are linearly independent if

$$\sum_{j=1}^{k} \lambda_j b_j = \mathbf{0} \Rightarrow (\forall j \in \{1, ..., k\} : \lambda_j = 0).$$

- $k > n$: solution can not be unique (more vectors than dimensions)
  - $k > n$ vectors of length $n$ can not be linearly independent
- $k = n$: $B_{mat}$ is square $\rightarrow$ apply determinant criterion
  - If $\det(B_{mat}) \neq 0$, unique solution $\lambda = B_{mat}^{-1}\mathbf{0} = \mathbf{0}$
- $k < n$: Bring $B_{mat}$ to general upper triangular form
  - Drop zero rows $\rightarrow$ square system, apply determinant criterion

# Eigenvalues

- **Eigenvalue** $\lambda \in \mathbb{R}$ of $A \in \mathbb{R}^{n \times n}$: $\exists x \in \mathbb{R}^n \backslash \{0\} : Ax = \lambda x$
  - $x \in \mathbb{R}^n \backslash \{0\}$ with $Ax = \lambda x$ is called an eigenvector of the eigenvalue $\lambda$
  - Eigenvectors are not unique: $Ax = \lambda x \Rightarrow A(cx) = \lambda(cx) \; \forall c \in \mathbb{R}$
  - Eigenspace of $\lambda$: $Span(\{x \in \mathbb{R}^n : Ax = \lambda x\})$
- How to find eigenvalues? Find roots of the characteristic polynomial $\mathcal{P}_A(\lambda)$

$$Eigenvalues(A) = \{\lambda \in \mathbb{R} : \mathcal{P}_A(\lambda) = 0\} \text{ where } \mathcal{P}_A(\lambda) = \det(A - \lambda \mathbf{I_n})$$

## Eigenvalues and Invertability

Let $A \in \mathbb{R}^{n \times n}$. Then, $A$ is invertable if and only if all eigenvalues of $A$ are non-zero.

*Proof.* $A$ is invertible if and only if $0 \neq \det(A) = \det(A - 0 \cdot \mathbf{I_n})$, which is the case if and only if $0$ is not an eigenvalue of $A$.

# Definiteness of Symmetric Matrices

## Definiteness of a Matrix

A **symmetric** square matrix $A \in \mathbb{R}^{n \times n}$ is called

- positive semi-definite if $\forall x \in \mathbb{R}^n : x'Ax \geq 0$
- negative semi-definite if $\forall x \in \mathbb{R}^n : x'Ax \leq 0$
- positive definite if $\forall x \in \mathbb{R}^n \backslash \{\mathbf{0}\} : x'Ax > 0$
- negative definite if $\forall x \in \mathbb{R}^n \backslash \{\mathbf{0}\} : x'Ax < 0$

Otherwise, it is called indefinite.

- Intuition: “smaller/greater than zero” equivalent for matrices

  $\Rightarrow$ Fundamentally important for optimization: sign of second derivative

# Definiteness, Eigenvalues and Invertability

## Definiteness and Eigenvalues

A symmetric square matrix $A \in \mathbb{R}^{n \times n}$ is

- positive (negative) definite if and only if all eigenvalues of $A$ are strictly positive (negative).
- positive (negative) semi-definite if and only if all eigenvalues of $A$ are strictly non-negative (non-positive).

## Definiteness and Invertability

If $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite or negative definite, it is invertible.

**Positive/negative definiteness is sufficient for invertability!**

# In-Class Exercises II

**Question 1** Consider the following matrices

$$A = \begin{pmatrix} 1 & 5 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 7 \\ 3 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 8 \\ 2 & 2 & 9 \end{pmatrix}$$

Calculate (i) $C'$ (ii) $2A$ (iii) $A + B$ (iv) $BC$ (v) $det(A)$ (vi) $A^{-1}$

**Question 2** Complete the illustratory example. What is the determinant of the original matrix?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} : \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & \end{array}\right)$$

# Recap Chapter 2

**Matrices** New mathematical structure

- Format with large practical applicability
- Way to systemize simultaneity

**Linear equation systems** Representation in matrix form as simplification

- Determinant criterion
- Alternative: Rank or eigenvalue criterion

**Invertibility**

- Gauß-Jordan Algorithm
- Directly linked to solution of linear equation systems

# That's all Folks!

**Please take a look at the problem set that we will discuss Monday morning.**

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