

Chapter 2: Linear Algebra II

Matrix Algebra

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Outline

In this chapter, we discuss

- The formal matrix concept and key definitions/types of matrices
- The matrix-based linear independence test
- Matrix inversion and its usefulness for solving equation systems
 - understand why matrix inversion works (elementary operations)
 - invert matrices using the Gauss-Jordan algorithm
- Key concepts related to matrices: rank, determinant, eigenvalues, definiteness, and how they are related

Motivation – Why Matrices?

- Formalize linear operations in Euclidean vector spaces
- Represent systems of linear equations in a simpler (and structured) manner
⇒ Simplify and regularize solution
- Derivatives of more complex functions "naturally" have matrix shape
- Typical (and practical) way to format empirical data
⇒ Central object of Econometrics

Matrices – Formal Definition

Matrix of dimension $n \times m$

Let $(a_{ij} : i \in \{1, \dots, n\}, j \in \{1, \dots, m\})$ be a collection of elements from a set X , i.e. $\forall i \in \{1, \dots, n\}, j \in \{1, \dots, m\} : a_{ij} \in X$. Then, the matrix A of these elements is the ordered two-dimensional collection

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix}$$

$\rightarrow \mathbb{R}^{n \times m}$

We call n the **row dimension** and m the **column dimension** of A . We write $A \in X^{n \times m}$, and $A = (a_{ij})_{i \in \{1, \dots, n\}, j \in \{1, \dots, m\}}$, or, if $n = m$, $A = (a_{ij})_{i, j \in \{1, \dots, n\}}$.

Matrices – Representation as a Vector Space

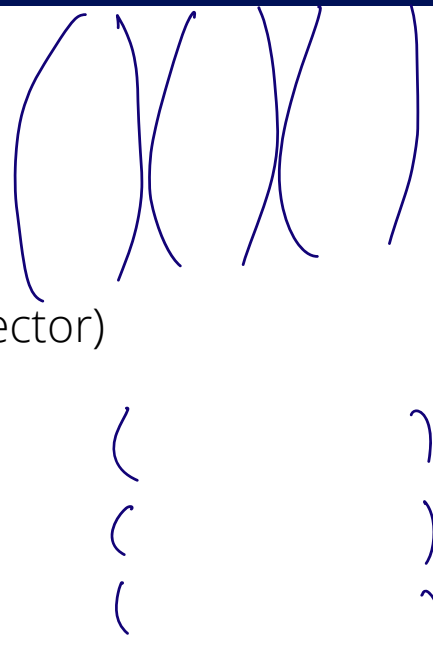
- Real Matrix ($X = \mathbb{R}$) as a **vector of vectors**:

- $\mathbf{a}_i^r = (a_{i1}, \dots, a_{im})$ i -th row of A (row vector)

- $\mathbf{a}_j^c = (a_{1j}, \dots, a_{mj})' = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$ j -th column of A (column vector)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^r \\ \vdots \\ \mathbf{a}_n^r \end{pmatrix} = (\mathbf{a}_1^c, \dots, \mathbf{a}_m^c)$$

- Typically: leave superscript r/c away if clear from context
- Vector space of real matrices: not vectors of real *numbers* (as $\mathbb{R}^n$), but vectors of real *vectors*!



Matrices – Basic Algebraic Operations

- Set of real $m \times n$ -Matrices denoted as $\mathcal{M}_{n \times m} = \{A \in \mathbb{R}^{n \times m}\}$

$\mathbb{R}^{n \times m}$
 $\rightarrow$ Vector Space

Addition of Matrices

For matrices A, B of **identical dimension**, i.e. $\exists n, m \in \mathbb{N} : A, B \in \mathcal{M}_{n \times m}$, their sum is obtained from addition of their elements, that is

$$\underline{A + B} = (a_{i,j} + b_{i,j})_{i \in \{1, \dots, n\}, j \in \{1, \dots, m\}} \quad \begin{pmatrix} 1 & 0 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} 7 & 2 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 8 & 8 \end{pmatrix}$$

Scalar Multiplication of Matrices

Let $A = (a_{i,j})_{i \in \{1, \dots, n\}, j \in \{1, \dots, m\}}$ and $\lambda \in \mathbb{R}$. Then, scalar multiplication of A with λ is defined element-wise, that is

$$\lambda A := (\lambda a_{i,j})_{i=1, \dots, n, j=1, \dots, m}$$

$$3 \cdot \begin{pmatrix} 1 & 0 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 15 \end{pmatrix}$$

Matrices – Basic Algebraic Operations

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} =$$

Multiplication of Matrices (Matrix Product)

Consider $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times k}$ where the column dimension of A is equal to the row dimension of B . Then, the matrix $C \in \mathbb{R}^{n \times k}$ of column dimension equal to the one of A and row dimension equal to the one of B is called the product of A and B , denoted $C = A \cdot B$, where

$$\forall i \in \{1, \dots, n\}, j \in \{1, \dots, k\} : c_{ij} = \sum_{l=1}^m a_{il} b_{lj}$$

$$\begin{matrix} A & B \\ n \times m & m \times k \\ B & A \\ m \times k & n \times m \end{matrix}$$

- Matrix Multiplication less straight forward
 - Dimensions of matrices have to match
 - Matrix multiplication is **not** commutative [even for square matrices $A \cdot B \neq B \cdot A$]

Matrices – Basic Algebraic Operations

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, B = \begin{pmatrix} b_1 & \cdots & b_k \end{pmatrix} \text{ (row/column notation). Then,}$$

$$AB = \begin{pmatrix} a'_1 \cdot b_1 & a'_1 \cdot b_2 & \cdots & a'_1 \cdot b_k \\ a'_2 \cdot b_1 & a'_2 \cdot b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a'_{n-1} \cdot b_k \\ a'_n \cdot b_1 & \cdots & a'_n \cdot b_{k-1} & a'_n \cdot b_k \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 4 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 + 3 \cdot 5 & 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 3 \\ 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 5 & 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 3 \\ 4 \cdot 1 + 4 \cdot 0 + 0 \cdot 5 & 4 \cdot 0 + 4 \cdot 1 + 0 \cdot 3 \end{pmatrix} = \begin{pmatrix} 16 & 11 \\ 0 & 1 \\ 4 & 4 \end{pmatrix}$$

Matrices – Key Properties of the Matrix Product

Associativity and Distributivity of the Product

Assuming that A, B, C are matrices of appropriate dimension, their product is

- (i) Associative: $(AB)C = A(BC)$ (order of multiplication is irrelevant!)
- (ii) Distributive over addition: $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$

Zero and Identity Matrix

Let $A \in \mathbb{R}^{n \times m}$. Then,

- (i) $A + \mathbf{0}_{n \times m} = A$. 
- (ii) For any $k \in \mathbb{N}$, $A \cdot \mathbf{0}_{m \times k} = \mathbf{0}_{n \times k}$ and $\mathbf{0}_{k \times n} \cdot A = \mathbf{0}_{k \times m}$. 
- (iii) For any $k \in \mathbb{N}$, $A \cdot \mathbf{I}_m = A$ and $\mathbf{I}_n \cdot A = A$. 

Matrices – Transposition

- Transposed matrix A' / A^T / A^t : swap of row and column index, e.g.

$$\begin{pmatrix} 1 & 0 & 4 \\ 3 & 1 & 2 \end{pmatrix}' = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 4 & 2 \end{pmatrix}$$

$$(1 \ 0 \ 4)' = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

Transposition, Sum, and Product

- (i) Let $A, B \in \mathbb{R}^{n \times m}$. Then, $(A + B)' = A' + B'$
- (ii) Let $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times k}$. Then, $(AB)' = B' A'$
- (iii) If $A \in \mathbb{R}^{1 \times 1}$, then A is actually a scalar and $A' = A$. (!?)

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Matrices – Special Matrices 1

- $A \in \mathbb{R}^{n \times m}$ is equal to the matrix B if (and only if) $B \in \mathbb{R}^{n \times m}$ and $\forall i, j: a_{ij} = b_{ij}$

$$\text{e.g. } \underline{\mathbf{0}_{2 \times 2}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underline{\mathbf{0}_{2 \times 3}}$$

- **Vector** of length n : (column) vector $\mathbf{a} \in \underline{\mathbb{R}^{n \times 1}}$, row vector $\mathbf{a} \in \underline{\mathbb{R}^{1 \times n}}$
- **Zero matrix**: $\mathbf{0}_{\underline{n \times m}} \in \mathbb{R}^{n \times m}$: $\mathbf{0}_{n \times m} = (a_{ij})_{i \in \{1, \dots, n\}, j \in \{1, \dots, m\}}$ where $\forall i, j: a_{ij} = 0$
- **Square** matrix: $A \in \mathbb{R}^{n \times m}$ with $n = m$, i.e. $A \in \mathbb{R}^{n \times n}$

$$\text{Example: } \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

Matrices – Special Matrices 2

- **Symmetric** matrix: square matrix $A \in \mathbb{R}^{n \times n}$ with $a_{ij} = a_{ji} \forall i, j$, e.g.

$$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \text{ but not } \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

$$A^T = A$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- **Diagonal** matrix: $A = (a_{ij})_{i,j \in \{1, \dots, n\}}$ with $(i \neq j \Rightarrow a_{ij} = 0)$, e.g.

$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$ We write $A = \text{diag}\{a_{11}, a_{22}, \dots, a_{nn}\}$

- **Identity** matrix: $I_n = \text{diag}\{1, 1, \dots, 1\}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Matrices – Special Matrices 3

Upper and Lower Triangular Matrix

A **square** matrix $A = (a_{ij})_{i,j \in \{1, \dots, n\}} \in \mathbb{R}^{n \times n}$ is said to be **upper triangular** if $(i > j \Rightarrow a_{ij} = 0)$, i.e. when the entry a_{ij} equals zero whenever it lies below the diagonal. Conversely, A is said to be lower triangular if A' is upper triangular, i.e. $(i < j \Rightarrow a_{ij} = 0)$.

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

→ Echelon-Form $\begin{pmatrix} 1 & -1 & 5 & 0 & 2 \\ 0 & 5 & 4 & 1 & 8 \\ 0 & 0 & 3 & 9 & -2 \end{pmatrix}$

and conversely $A' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & -4 & 0 \\ 4 & 0 & 3 & 2 \end{pmatrix}$.

Matrices – Matrix Vector Space

- Linear Operations: matrix addition and scalar multiplication as defined before
⇒ follows regular rules except for (not required) matrix product commutativity
- Existence of multiplicative identity (1), additive identity ($\mathbf{0}_{n \times m}$) and inverse ($-A$)

- Maximum Norm of $(\mathcal{M}_{n \times m}, +, \cdot)$:

$$\|A\|_{\infty} = \max\{|a_{ij}| : i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$$

1. $\|A\|_{\infty} \geq 0$ and $(\|A\|_{\infty} = 0 \Leftrightarrow \forall i, j: a_{ij} = 0, \text{ i.e. } A = \mathbf{0}_{n \times m})$
2. Triangle inequality transfers from $|\cdot|$
3. Absolute homogeneity follows from $|\cdot|$ as well

⇒ Can define norm-induced metrics, convergence, etc. like for $\mathbb{R}^n$!

Linear Equation Systems – Matrix Representation

- Translate linear equation system into matrix notation: $1x_1 + 2x_2 + 3x_3$

$$\begin{array}{rrcrcl} 1 \cdot x_1 & + & 2 \cdot x_2 & + & 3 \cdot x_3 & = & 2 \\ 0 \cdot x_1 & + & 1 \cdot x_2 & + & 4 \cdot x_3 & = & 2 \\ 5 \cdot x_1 & + & 9 \cdot x_2 & + & 0 \cdot x_3 & = & 3 \end{array} \iff \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{=x} = \underbrace{\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}}_{=b}$$

$$\begin{aligned} ax &= b \\ x &= \frac{b}{a} \\ x &= b a^{-1} \end{aligned}$$

- Interpretation: search of arguments x that solve $f_A(x) = b$ for $f_A(x) = Ax$
 $\Rightarrow$ how to use A to characterize the solutions and determine them?
- Easiest scenarios: “square system” – as many equations as unknowns
- Simplest of all: one equation in one unknown: $f_a(x) = \underline{ax} = b, a, b \in \mathbb{R}$
 - If $a \neq 0$, then unique solution $x = a^{-1}b = b/a$
 - If $a = 0$ and $b = 0$: infinitely many solutions $x \in \mathbb{R}$
 - If $a = 0$ and $b \neq 0$: no solution (“ b never reached by $a \cdot x$ ”)

Linear Equation Systems – Solution Idea

- 1×1 system: generalization step

- Unique solution if and only if a is invertible [$\iff a \neq 0$], i.e.

$$\exists a^{-1} \in \mathbb{R} : a^{-1}a = aa^{-1} = 1$$

$$\rightarrow x^* = a^{-1}b$$

- Otherwise: infinitely many/no solutions
- For $n \times n$ systems ($Ax = b$):
 - Unique solution if and only if A is invertible, i.e.

$$\exists A^{-1} \in \mathbb{R}^{n \times n} : A^{-1}A = AA^{-1} = I_n$$

$$\rightarrow x^* = a^{-1}b$$

- Otherwise: infinitely many/no solutions
- Now: when does an inverse A^{-1} exist?

$$ax = b$$

$$\underbrace{a^{-1}a}_1 x = a^{-1}b$$

$$x = a^{-1}b$$

$$Ax = b$$

$$\underbrace{A^{-1}A}_I x = A^{-1}b$$

$$1_n x = x = A^{-1}b$$

Matrix Inversion – Definition and Uniqueness

$$(A^{-1})^{-1} = A$$

- Inverse matrix of square matrix $A \in \mathbb{R}^{n \times n}$: $A^{-1} \in \mathbb{R}^{n \times n}$ so that

$$A^{-1}A = AA^{-1} = I_n$$

- The inverse matrix is unique!

Proof. Let $B, C \in \mathbb{R}^{n \times n}$ with $BA = AB = I_n$ and $CA = AC = I_n$. Then, it holds that

$$C = CI_n = C(AB) = \underbrace{(CA)}_{=I_n} B = I_n B = B,$$

$$\text{i.e. } C = B$$

- How to (dis)prove invertability?
- How to find the inverse?

B, C inverses of A

$$BA = AB = I_n$$

$$CA = AC = I_n$$

$$B, C \in \mathbb{R}^{n \times n}$$

$$\underline{B} = B I_n = B(AC)$$

$$= (BA) C = I_n C = \underline{C}$$

$$3 \rightarrow \frac{1}{3}$$

$$a^{-1}a = aa^{-1}$$

$$3 \cdot \frac{1}{3}$$

$$\frac{1}{3} \cdot 3$$

Definition $A^{-1} : A^{-1}A = I_n$

$$\begin{aligned} (A^{-1})^{-1} A^{-1} &= I_n \\ &= A \end{aligned}$$

$$A^{-1}A = I_n = AA^{-1}$$

Matrix Inversion – The Gauß-Jordan Algorithm

- Process to find inverse by converting a (square) matrix into identity matrix

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}}_A \underbrace{\left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right.}_{I_3} \longrightarrow \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \underbrace{\left| \begin{array}{ccc} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{array} \right.}_{A^{-1}}$$

- Allowed operations:

1. Interchange of (two) rows
2. Scalar multiplication of a row
3. Adding (scaled) rows to other rows

Handwritten notes:

$$\begin{aligned} x_1 &= \\ x_2 &= \\ x_3 &= \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= \\ -2x_1 &+ x_3 \\ x_1 - x_2 &= \end{aligned}$$

Matrix Inversion – Elementary Matrix Operations

Elementary Matrix Operations

Let A with rows $a'_1, \dots, a'_n \in \mathbb{R}^n$ and $\tilde{A}$ with rows $\tilde{a}'_1, \dots, \tilde{a}'_n \in \mathbb{R}^n$ be two square matrices. The three elementary matrix operations are mappings $A \rightarrow \tilde{A}$ given by

(E1) Interchange of two rows i, j : $\tilde{a}_i = a_j$, $\tilde{a}_j = a_i$ and $\tilde{a}_k = a_k \forall k \notin \{i, j\}$,

(E2) Scalar multiplication of a row i with $\lambda \neq 0$: $\tilde{a}_i = \lambda a_i$ and $\tilde{a}_j = a_j \forall j \neq i$,

(E3) Addition of a scaled row j to row i : $\tilde{a}_i = a_i + \lambda a_j$, $\lambda \in \mathbb{R}$, and $\tilde{a}_j = a_j \forall j \neq i$.

- Formalizes the operation used in Gauß-Jordan's algorithm
- Not to be confused with **basis** operations + and ·

$$E_1 A = \tilde{A}$$

$$A \rightarrow \tilde{A}$$

$$\rightarrow E_1 \quad E_2 \quad E_3$$

Matrix Inversion – Formalizing Gauß-Jordan

- Elementary Operations represented by **left-multiplication**: $\tilde{A} = EA = E_2 E_1 A$
- Transforming A to I_n (in k steps): $E_k E_{k-1} \dots E_1 A = I_n$
 $\Rightarrow$ Recall definition of A^{-1} : $A^{-1}A = I_n$ by uniqueness: $A^{-1} = E_k E_{k-1} \dots E_1$
 $\Rightarrow$ Nothing special about E_i (except relatively easy to find)

Example:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & ? & ? & ? \\ 0 & 1 & 0 & ? & ? & ? \\ 0 & 0 & 1 & ? & ? & ? \end{array} \right)$$

$$\underbrace{E_3 E_2 E_1}_{A^{-1}} A = I_n$$

$$\underbrace{E_3 E_2 E_1}_{A^{-1}} I_n = A^{-1}$$

Matrix Inversion – Helpful Theorems

Invertability: Transpose and Product

Suppose that $A, B \in \mathbb{R}^{n \times n}$ are invertible. Then,

- A' is invertible and $(A')^{-1} = (A^{-1})'$,
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$,
- $\forall \lambda \in \mathbb{R}, \lambda \neq 0, \lambda A$ is invertible and $(\lambda A)^{-1} = 1/\lambda A^{-1}$.

$$(AB)^{-1}AB = I_n$$

$$B^{-1}A^{-1}AB = B^{-1}B = I_n$$

$$(2A)^{-1} = \frac{1}{2}A^{-1}$$

Inverse of a 2×2 Matrix

Consider a 2×2 matrix A . Then, as long as $ad \neq bc$, for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinant – Definition

- Submatrix A_{-ij} of A : Matrix A without row a_i^r and column a_j^c

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow A_{-23} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

Diagram illustrating the Laplace expansion of a determinant along a row:

$$\det \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} \rightarrow \det \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$$

Diagram illustrating the Laplace expansion of a determinant along a column:

$$\det \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} \rightarrow \det \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$$

Determinant

Let $A \in \mathbb{R}^{n \times n}$. Then, we define the determinant of A , denoted $\det(A)$ or $|A|$, for all $n \in \mathbb{N} : n \geq 2$ as

$$\det(A) := \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{-ij}) \text{ for } i=1$$

Handwritten calculation for $\det(A)$ using the first row:

$$\det(A) = 1 \cdot 1 \det \begin{pmatrix} 2 & 4 \\ 0 & 4 \end{pmatrix} + (-1) \cdot 2 \cdot \det \begin{pmatrix} 0 & 4 \\ 0 & 4 \end{pmatrix} + 1 \cdot 3 \det \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$= 1 \det \begin{pmatrix} 2 & 4 \\ 0 & 4 \end{pmatrix} - 2 \det \begin{pmatrix} 0 & 4 \\ 0 & 4 \end{pmatrix} + 3 \det \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Otherwise, for $n = 1$, i.e., $A = (a_{11})$, it is given by $\det(A) = a_{11}$.

Determinant – Laplace Expansion Theorem

- **Laplace Expansion Theorem:** It holds that
 - $\det(A) := \sum_j (-1)^{i+j} a_{ij} \det(A_{-ij})$ for arbitrary $i \in \{1, \dots, n\}$ (expansion by i -th row)
 - $\det(A) = \sum_i (-1)^{i+j} a_{ij} \det(A_{-ij})$ for arbitrary $j \in \{1, \dots, n\}$ (expansion by j -th column)
- ⇒ Look for rows/columns with many zeros!
- ⇒ Typically, do Laplace expansions until down to 3×3 submatrices

- Determinant of “small” matrices:

1. If $n = 2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(A) = ad - bc$.

2. If $n = 3$ and $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then $\det(A) = \underline{aei} + \underline{bfg} + \underline{cdh} - (\underline{ceg} + \underline{bdi} + \underline{afh})$.

Determinant – Triangular Matrices

- The determinant of triangular matrices is equal to its **diagonal product**

$$\det(A) = a_{11} \det \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} = \underline{a_{11} a_{22}} \det \begin{pmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{43} & \cdots & a_{4n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

$$= \dots = \prod_{i=1}^n a_{ii}$$

- $\det(A) \neq 0 \Leftrightarrow (\forall i \in \{1, \dots, n\} : a_{ii} \neq 0)$
- Triangular A is invertible if and only if $\det(A) \neq 0$
- This holds also for non-triangular A

$\rightarrow$ Determinant - Criterion

LSE $\rightarrow$ unique solution
 $\Leftrightarrow \det(A) \neq 0$

Determinant – Elementary Operations

Determinant and Elementary Operations

Let $A \in \mathbb{R}^{n \times n}$ and $\tilde{A}$ the resulting matrix for the respective EO. Then,

1. for (E1) (interchange of two rows), we have $\det(\tilde{A}) = -\det(A)$
2. for (E2) (row multiplication with a scalar $\lambda \neq 0$), $\det(\tilde{A}) = \lambda \det(A)$
3. for (E3) (addition of multiple of row to another row), $\det(\tilde{A}) = \det(A)$

- For any square matrix A :

- We can use EOs to bring A to a triangular matrix $\tilde{A}$
- We can use EOs to bring $\tilde{A}$ to I_n if $\det(\tilde{A}) \neq 0$
- EOs do not affect the property “ $\det \neq 0$ ”! Thus, $\det(\tilde{A}) \neq 0$ if and only if $\det(A) \neq 0$

$\rightarrow \det \neq 0$

Determinant – Helpful Properties

Determinant and Product Rule

Let $A, B \in \mathbb{R}^{n \times n}$ be two square matrices of equal dimension. Then,

- $\det(AB) = \det(A) \det(B)$

- $\det(A^{-1}) = 1/\det(A)$

- $\det(\lambda A) = \lambda^n \det(A)$ for all $\lambda \in \mathbb{R}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \det = 1$$

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \rightarrow \det \lambda^n$$

- So far: " $\det(A) \neq 0 \Rightarrow A$ invertible" [$\rightarrow$ product rule allows to show converse]

$\Rightarrow$ **we can invert A if and only if $\det(A) \neq 0$!**

$\Rightarrow$ Square systems $Ax = b$: **unique solution** $x^* = A^{-1}b \Leftrightarrow \det(A) \neq 0$

A square matrix is invertible if and only if it has non-zero determinant! In a Square System, we (almost) always use the determinant invertability criterion!

Matrix Rank – An Alternative Solution Criterion

• **Column rank:** Number of LI columns of A

• **Row rank:** Number of LI rows of A

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$\Rightarrow$ Theorem: Column rank = row rank =: **Rank**, denoted $\text{rk}(A)$ or $\text{rk } A$

[Corollary: rank bound $\text{rk } A \leq \min\{m, n\}$ for $n \times m$ matrix A]

• For square matrices ($m = n$), A has **full rank** if $\text{rk}(A) = n$

• **The rank is unchanged by elementary operations (E1) to (E3)!**

Rank Criterion for Linear Equation Systems

Let $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. Then, the system $Ax = b$ has a unique solution if and only if $\text{rk}(A) = n$.

full rank $\Leftrightarrow$ unique solution

Solvability of Linear Equation Systems – Summary

- Summarizing all we've seen so far:

Invertability, Rank and Determinant of Square Matrices

Consider **square** matrix $A \in \mathbb{R}^{n \times n}$. Then, the following statements are equivalent:

1. A is invertible,

2. $\det(A) \neq 0$,

3. $\text{rk } A = n$,

4. for any $b \in \mathbb{R}^n$, $Ax = b$ has a unique solution for x ,

5. Any triangular matrix $\tilde{A}$ obtained from applying EOs to A has only non-zero diagonal entries.

column
row vectors A are lin. indep.

$$\Leftrightarrow A^{-1} b = x$$

Non-Square Equation Systems

- Underidentified system of equations
 $\Rightarrow n < m$, strictly less equations than unknowns
 - Information deficit $\rightarrow$ solution has degrees of freedom (not fully pinned down)
- Overidentified system of equations
 $\Rightarrow n > m$, strictly more equations than unknowns
 - Information contradiction $\rightarrow$ no solution can exist
 - Information redundancy $\rightarrow$ system of equation can be reduced

$$\begin{array}{l} 3x = 1 \\ 6x = 2 \end{array} \quad \begin{array}{l} 1 \\ 5 \end{array}$$

Generalized Rank Condition

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then, the system $Ax = b$ has a unique solution if and only if $b \in \text{Co}(A) := \text{Span}(a_1^c, \dots, a_n^c)$ and $\text{rk}(A) = n$.

Column space $\text{Co}(A)$: space spanned by the columns of A

Testing Linear Independence using Matrices

- Recall from yesterday:

Testing Linear Independence

The set of vectors $B = \{b_1, b_2, \dots, b_k\}$ are linearly independent if

$$\sum_{j=1}^k \lambda_j b_j = \mathbf{0} \Rightarrow (\forall j \in \{1, \dots, k\} : \lambda_j = 0).$$

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \lambda_1 = 0$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

- $\sum_{j=1}^k \lambda_j b_j = \mathbf{0}$ is a linear equation system!

- Matrix form: $B_{mat} \lambda = \mathbf{0}$ with $\lambda = (\lambda_1, \dots, \lambda_k)'$, $B_{mat} = (b_1, b_2, \dots, b_k)$

- We need (i) a unique solution (ii) equal to $\lambda = \mathbf{0}$ $\rightarrow$ lin. indep.

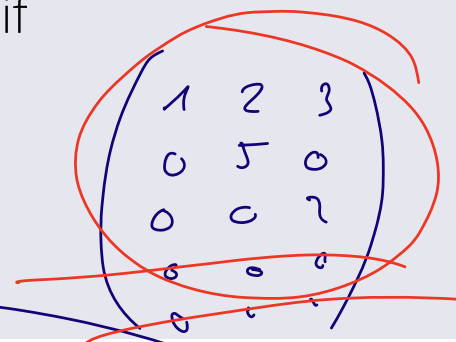
$\rightarrow$ determinant criterion

Testing Linear Independence using Matrices

Testing Linear Independence

The set of vectors $B = \{b_1, b_2, \dots, b_k\}$ are linearly independent if

$$\sum_{j=1}^k \lambda_j b_j = \mathbf{0} \Rightarrow (\forall j \in \{1, \dots, k\} : \lambda_j = 0).$$



- $k > n$: solution can not be unique (more vectors than dimensions)
 - $k > n$ vectors of length n can not be linearly independent
- $k = n$: B_{mat} is square $\rightarrow$ apply determinant criterion
 - If $\det(B_{mat}) \neq 0$, unique solution $\lambda = B_{mat}^{-1} \mathbf{0} = \mathbf{0}$
- $k < n$: Bring B_{mat} to general upper triangular form $\rightarrow$ Echelon form
 - Drop zero rows $\rightarrow$ square system, apply determinant criterion

Eigenvalues & eigenvectors

→ Polynomial of order n has exactly n complex roots

$$A \otimes x = \lambda x$$

- **Eigenvalue** $\lambda \in \mathbb{R}$ of $A \in \mathbb{R}^{n \times n}$: $\exists x \in \mathbb{R}^n \setminus \{0\} : Ax = \lambda x$
 - $x \in \mathbb{R}^n \setminus \{0\}$ with $Ax = \lambda x$ is called an **eigenvector** of the eigenvalue λ
 - Eigenvectors are not unique: $Ax = \lambda x \Rightarrow A(cx) = \lambda(cx) \forall c \in \mathbb{R}$
 - **Eigenspace** of λ : $\text{Span}(\{x \in \mathbb{R}^n : Ax = \lambda x\})$
- How to find eigenvalues? Find roots of the **characteristic polynomial** $\mathcal{P}_A(\lambda)$

$$\text{Eigenvalues}(A) = \{\lambda \in \mathbb{R} : \mathcal{P}_A(\lambda) = 0\} \text{ where } \mathcal{P}_A(\lambda) = \det(A - \lambda I_n)$$

$$\det(A - \begin{pmatrix} \lambda & a & c \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}) = 0$$

Eigenvalues and Invertability

Let $A \in \mathbb{R}^{n \times n}$. Then, A is invertible if and only if all eigenvalues of A are non-zero.

Proof. A is invertible if and only if $0 \neq \det(A) = \det(A - 0 \cdot I_n)$, which is the case if and only if 0 is not an eigenvalue of A .

Definiteness of Symmetric Matrices

$$x \in \mathbb{R} \begin{cases} \rightarrow x > 0 \\ \rightarrow x < 0 \end{cases}$$

$$(x) y^2 > 0$$

Definiteness of a Matrix

A **symmetric** square matrix $A \in \mathbb{R}^{n \times n}$ is called

- positive semi-definite if $\forall x \in \mathbb{R}^n : x'Ax \geq 0$
- negative semi-definite if $\forall x \in \mathbb{R}^n : x'Ax \leq 0$
- positive definite if $\forall x \in \mathbb{R}^n \setminus \{0\} : x'Ax > 0$
- negative definite if $\forall x \in \mathbb{R}^n \setminus \{0\} : x'Ax < 0$

Otherwise, it is called indefinite.

$$x'Ax \geq 0$$

(semi)positive definite

$$\leq 0$$

(semi)negative definite

- Intuition: “smaller/greater than zero” equivalent for matrices
 $\Rightarrow$ Fundamentally important for optimization: sign of second derivative

Definiteness, Eigenvalues and Invertability

Definiteness and Eigenvalues

A symmetric square matrix $A \in \mathbb{R}^{n \times n}$ is

- positive (negative) definite if and only if all eigenvalues of A are strictly positive (negative).
- positive (negative) semi-definite if and only if all eigenvalues of A are strictly non-negative (non-positive).

Definiteness and Invertability

$\rightarrow$ invertible $\rightarrow$ invertible

If $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite or negative definite, it is invertible.

Positive/negative definiteness is sufficient for invertability!

In-Class Exercises II

Question 1 Consider the following matrices

$$A = \begin{pmatrix} 1 & 5 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 7 \\ 3 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 8 \\ 2 & 2 & 9 \end{pmatrix}$$

$$\begin{aligned} B \cdot C &= \begin{pmatrix} 2 & 7 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 8 \\ 2 & 2 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 2 \cdot 1 + 7 \cdot 2 & 2 \cdot 0 + 7 \cdot 2 & 2 \cdot 8 + 7 \cdot 9 \\ 3 \cdot 1 + 1 \cdot 2 & 3 \cdot 0 + 1 \cdot 2 & 3 \cdot 8 + 1 \cdot 9 \end{pmatrix} \\ &= \begin{pmatrix} 16 & 14 & 79 \\ 5 & 2 & 33 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{5} & -\frac{1}{10} \end{pmatrix} \end{aligned}$$

Calculate (i) C' (ii) $2A$ (iii) $A + B$ (iv) BC (v) $\det(A)$ (vi) $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{-10} \begin{pmatrix} 0 & -5 \\ -2 & 1 \end{pmatrix}$

Question 2 Complete the illustrative example. What is the determinant of the original matrix?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} : \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{array} \right)$$

$$\begin{array}{c} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \\ \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ \textcircled{5} & 6 & 0 & 0 & 0 & 1 \end{array} \right) \end{array}$$

$\begin{matrix} -5 & -10 & -15 \\ -5 & 0 & 0 \end{matrix}$
 $-5 \cdot \text{I}$

$$\left(\begin{array}{ccc|ccc} 1 & \textcircled{2} & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & \textcircled{-4} & -15 & -5 & 0 & 1 \end{array} \right)$$

$\begin{matrix} -2 & -8 \\ 4 & 16 \end{matrix}$
 $\begin{matrix} -2 \cdot \text{I} \\ +4 \cdot \text{I} \end{matrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \textcircled{-5} & 1 & -25 & -20 \\ 0 & 1 & \textcircled{4} & 0 & 20 & 16 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array} \right)$$

$\begin{matrix} 5 \\ -4 \end{matrix}$
 $\begin{matrix} +5 \cdot \text{III} \\ -4 \cdot \text{III} \end{matrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array} \right)$$

$\underbrace{\hspace{10em}}_{\text{I}_n} \quad \underbrace{\hspace{10em}}_{A^{-1}}$

Recap Chapter 2

Matrices New mathematical structure

- Format with large practical applicability
- Way to systemize simultaneity

Linear equation systems Representation in matrix form as simplification

- Determinant criterion
- Alternative: Rank or eigenvalue criterion

Invertibility

- Gauß-Jordan Algorithm
- Directly linked to solution of linear equation systems

That's all Folks!

Please take a look at the problem set that we will discuss Monday morning.

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