E600 Mathematics Fall Semester 2026

Problem Set 1

Linear Algebra I: Vector Spaces, Metrics, Set Properties

Exercise 1: Key Concepts of Vector Spaces

a.) Linear independence

1. For which real numbers α are the two vectors $\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix}$ linearly independent?

2. For such an α, find a third vector to form a basis of $\mathbf{R}^3$.

b.) Scalar Products

Compute $a \cdot b$ for

1. $a = (2,5,1)'$, $b = (1,1,3)' \in \mathbb{R}^3$

2. $a = (2,0,-3,4)$, $b = (9,-8,7,-6) \in \mathbb{R}^4$

What do you tell your colleague who claims to have found $v \in \mathbb{R}^n$ so that $v \cdot v = -1$?

c.) The Binary Metric

Consider a real vector space $\mathbb{X} = (X, +, \cdot)$, and define the binary metric

$$d_B : X \times X \mapsto \mathbb{R}, d_B(x,y) = \mathbb{1}[x \neq y]$$

Show that the function indeed constitutes a metric, i.e. show that it satisfies the three properties that define a metric function.
In case you are not familiar with the notation used here, $\mathbb{1}[S(x)]$ is a so-called indicator function for a statement $S(x)$ related to x that takes the value 1 when $S(x)$ is true and 0 otherwise. Accordingly,

$$d_B(x,y) = \mathbb{1}[x \neq y] = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

d.) Norm-Induced Metric

Let $\|.\|$ be a norm on the vector space X. Show that $d(x,y) := \|x - y\|$ is a metric on X.

Exercise 2: Testing for Set Properties

In this exercise, we practice the investigation of key set properties discussed in lecture 1. As set properties refer to metric or normed vector spaces, respectively, you need to know which distance measure to consider. You can assume (as always when nothing else is explicitly specified) that we deal with Euclidean spaces, but feel free to choose any other p-norm (including $p = \infty$) that may work better for you.

a.) A subset of the real line

Let $S_1 := \{x \in \mathbb{R} : x^2 \leq 4\}$. Is this set open, closed, compact and/or convex?

b.) Two dimensions

Let $S_2 := \{x \in \mathbb{R}^2 : x_1 \leq 3\}$. Is this set open, closed, compact and/or convex?

Exercise 3: Vector Spaces

Take arbitrary $x, y, z \in \mathbb{R}$ and

$$V = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}$$

1. Show that $V \neq \emptyset$.
2. Show that V is closed under vector addition, i.e. $\forall v_1, v_2 \in V : v_1 + v_2 \in V$.
3. Show that V is closed under scalar multiplication, i.e. $\forall v \in V, \alpha \in \mathbb{R} : \alpha v \in V$.

Exercise 4: Basis of a Vector Space

a.) Bases of the $\mathbb{R}^2$

Which of the following sets are bases of the $\mathbb{R}^2$ (S&B, Ex. 11.12)?

$$S_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix} \right\}, \quad S_3 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\}, \quad S_4 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, e_2 \right\}$$

b.) A Basis of a Function Space

Consider the set of second order polynomials,

$$\mathbb{P}_2(X) := \{f : X \mapsto \mathbb{R} : (\exists a, b, c \in \mathbb{R} : f(x) = ax^2 + bx + c)\}$$

It turns out to be the case that $\mathbb{P}_2(X)$ is a subspace of $\mathbb{F}(X, \mathbb{R})$, the space of all functions mapping from X to $\mathbb{R}$. Find a basis for $\mathbb{P}_2(X)$, i.e. a set of functions $f_1, f_2, \ldots, f_k$ such that (1) every second order polynomial can be written as a linear combination of these functions and (2) the functions are all linearly independent, i.e. they can not be written as linear combinations of each other.