E600 Mathematics Fall Semester 2026

Problem Set 2

Linear Algebra II: Matrix Algebra

Exercise 1: Matrix Multiplication

a.) Two Matrices

Determine whether the following matrices exist, and if so, compute them: AB, $B'A'$ and BA for

$$A = \begin{pmatrix} 0 & 2 \\ 3 & -5 \\ -2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{pmatrix}$$

Hint: Be aware of the rules for transposition and matrix operations to take some shortcuts!

$$A_{3\times2} \, B_{2\times3} = C_{3\times3}$$

$$AB = \begin{pmatrix} 0\cdot(-1)+2\cdot4 & 0-10 & 0+12 \\ -3-20 & 6+25 & -9-30 \\ 2+12 & -4-15 & 6+18 \end{pmatrix} = \begin{pmatrix} 8 & -10 & 12 \\ -23 & 31 & -39 \\ 14 & -19 & 24 \end{pmatrix}$$

$$B'A' = (AB)' = \begin{pmatrix} 8 & -23 & 14 \\ -10 & 31 & -19 \\ 12 & -39 & 24 \end{pmatrix}$$

$$B_{2\times3} \, A_{3\times2} = C_{2\times2}$$

$$BA = \begin{pmatrix} 0+6+6 & -2-10-9 \\ 0-15-12 & 8+25+18 \end{pmatrix} = \begin{pmatrix} 12 & -21 \\ -27 & 51 \end{pmatrix}$$

$\in \mathbb{R}^{2\times 3}$

b.) Some more Products

Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -5 & 3 \\ 2 & 4 \end{pmatrix}.$$

Determine whether the following matrices exist, and if so, compute them:

1. AB 2. BA 3. $BA + C$ 4. $AB + C$

$$AB = \begin{pmatrix} 1+0+4 & 1+0+0 \\ 2+6-4 & 2-3+0 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 4 & -1 \end{pmatrix} + \begin{pmatrix} -5 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 6 & 3 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1+2 & 0+3 & -1+1 \\ 2-2 & 0-3 & -2-1 \\ -4+0 & 0+0 & 4+0 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 0 \\ 0 & -3 & -3 \\ -4 & 0 & 4 \end{pmatrix} + \begin{pmatrix} -5 & 3 \\ 2 & 4 \end{pmatrix} \text{ not def.}$$

Column vectors

c.) Right-Multiplication of Vectors and Dimensionality

Let A be the matrix as in b.). What $n \in \mathbb{N}$ must we choose so that $x \in \mathbb{R}^n$ can be right-multiplied to A, i.e. as Ax? What about $A'x$?

$A \in \mathbb{R}^{2 \times 3}$

$x \in \mathbb{R}^{3 \times 1}$

Ax

$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \cdot \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \to \in \mathbb{R}^3$

$A' \in \mathbb{R}^{3 \times 2}$

$x \in \mathbb{R}^{2 \times 1}$

$A'x$

$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \Rightarrow \in \mathbb{R}^2$

Exercise 2: Elementary Matrix Operations

Here, we convince ourselves again that the elementary operations really work in the way we introduced them: Consider the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

$$\underbrace{E \cdots E\, E_1}_{A^{-1}} A = I_n \qquad \to h = A^{-1}$$

Define the matrix E so as to

1. interchange rows 2 and 3 (call the matrix E_1),
2. multiply rows 1 and 3 with $\lambda = 5 \neq 0$ (call the matrix E_2),
3. subtract two times row 1 from row 2 (call the matrix E_3).

Multiply out EA for E_3 and check that indeed, the respective operation is performed.

$$E \cdot I_n = E \qquad E_1$$

1. $$E \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2 \leftrightarrow 3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= E \qquad =$$

$$(A \mid I_n)$$

2. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[5 \cdot \mathrm{III}]{5 \cdot \mathrm{I}} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} = E_2 \qquad (I_n \mid A^{-1})$$

3. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\mathrm{II}' = \mathrm{II} - 2 \cdot \mathrm{I}} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_3$$

Exercise 3: Determinant, Definiteness and Eigenvalues

a.) Determinant Rules

invertable $\Leftrightarrow$ det $\neq 0$

For the following matrices, compute the determinant using an appropriate rule.

1. $A = \begin{pmatrix} 3 & 8 \\ 2 & -1 \end{pmatrix}$ $\rightarrow$ $\det A = 3 \cdot (-1) - 2 \cdot 8 = -19$

2. $B = \begin{pmatrix} -3 & 2 & 4 \\ -6 & 5 & 4 \\ 1 & -1 & 0 \end{pmatrix}$ $\rightarrow$ $\det B = -3 \cdot 5 \cdot 0 + 2 \cdot 4 \cdot 1 + 4(-6)(-1) - 1 \cdot 5 \cdot 4 - 2(-6) \cdot 0 - (-3) \cdot 4 \cdot (-1) =$
$= 0 + 8 + 24 - 20 + 0 - 12 = 0$

3. $C = \begin{pmatrix} 0 & 0 & 2 \\ 3 & 1 & -1 \\ 2 & 2 & 4 \end{pmatrix}$

Hint: You can test your understanding of the Laplace method by using an appropriate expansion at 3. (of course, the 3×3 rule is still perfectly fine here as well).

$\rightarrow (-1)^{i+j}$

$\det C = +0 \cdot \det \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} - 0 \cdot \det \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$
$+ 2 \cdot \det \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} = 2(6-2) = 8$

Alternative: $2 \cdot \det \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} = 2 \left[3 \det(2) - 1 \det(2) \right]$
$= 2 \left[3 \cdot 2 - 1 \cdot 2 \right] = 8$

b.) Definiteness → "Matrix-Equivalent of $\gtrless 0$"

Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

Is A positive or negative (semi-)definite? What can you say about the invertability of A from this fact?

p.d. $\quad \forall x \in \mathbb{R}^3,\ x \neq 0 \quad x'Ax > 0$

p. sd. $\quad x'Ax \geq 0$

$$(x_1 \ x_2 \ x_3)\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (2x_1 - x_3 \quad x_1 + 3x_2 + 1x_3 \quad x_2 - x_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= (2x_1 - x_3)x_1 + \underbrace{(x_1 + 3x_2 + x_3)x_2}_{x_1x_2 + 3x_2^2 + x_2x_3} + \underbrace{(x_2 - x_3)x_3}_{+x_2x_3 - x_3^2}$$

$$= 2x_1^2 - x_1x_3 + x_1x_2 + 3x_2^2 + 2x_2x_3 - x_3^2 \gtrless 0$$

$x^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ → $2 + 0 > 0$

$x^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ → $0 - 1 < 0$

→ indefinite

Sylvester's Theorem

det > 0   det > 0   det < 0

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

det > 0 $\Leftrightarrow$ A positive definite

1. det < 0
2. det > 0
3. det < 0
⋮

$\Leftrightarrow$ A negative definite

→ positive definite

det > 0   det > 0   det > 0

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$x'Ax \to \underset{\geq 0}{x_1^2} + \underset{\geq 0}{(x_1 - x_2)^2} + \underset{\geq 0}{(x_2 - x_3)^2} + \underset{\geq 0}{x_3^2}$$

> 0

→ positive definite

All Eigenvalues > 0 → positive definite

All Eigenvalues < 0 → negative definite

Exercise 4: Matrix Inversion

For the following matrices, perform a test for invertability (using e.g. the determinant) and, if possible, compute the inverse matrix, using either a shortcut theorem or the Gauss-Jordan method:

1. $A = \begin{pmatrix} 3 & 8 \\ 2 & -1 \end{pmatrix}$ $\Rightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{-19}\begin{pmatrix} -1 & -8 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1/19 & 8/19 \\ 2/19 & -3/19 \end{pmatrix}$

2. $B = \begin{pmatrix} -3 & 2 & 4 \\ -6 & 5 & 4 \\ 1 & -1 & 0 \end{pmatrix}$ $\Rightarrow \det = 0$ ✗

3. $C = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\det C = 3 - 2 = 1$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 2_{-2} & 1_{-1} & 3_{-2} & 1 & 0_{-2} & 0_{-1} \end{array}\right) \quad -2\cdot\mathrm{I} \quad -\mathrm{II}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1_{-1} & 0_{-1} & 1_{+2} & 0_{+1} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right) \quad -\mathrm{III}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -1 & 3 & 1 \\ 0 & 0 & 1 \\ 1 & -2 & -1 \end{pmatrix}$$

Exercise 5: Eigenvalues, Definiteness and Invertability

Let $A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$.

$\to$ Eigenvectors

Determine the eigenvalues of A. What can you say about the definitness of A? Is A invertible? How could you have checked invertability more directly?

$x \neq 0$ $\quad Ax = \lambda x$ $\quad \to$ roots of the characteristic Polynomial of A

$$Ax - \lambda x = 0 \quad \to \text{0-vector}$$

$$Ax - \lambda \underbrace{I_n x}_{=x} = \underbrace{(A - \lambda I_n)}_{\text{Matrix} = P} x = 0 \quad \to P^{-1}$$

$\det(P) = 0$ $\qquad \det(A - \lambda I_n)$ $\qquad Px = 0$ System of linear Equations

$$\det\left(\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \lambda I_n\right) = \det\left(\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

$$\det\begin{pmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{pmatrix} = \underbrace{(3-\lambda)(1-\lambda) - 8}_{\text{Characteristical Polynomia}}$$

$x = 0$ arbitrary λ

$$3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0 \qquad \lambda = \frac{4}{2} \pm \sqrt{\left(\frac{4}{2}\right)^2 + 5} = 2 \pm \sqrt{4+5} = 2 \pm 3$$

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 5$$

$\to$ neither ~~positive~~ nor negative definite

$$A\underline{x} = \lambda x \qquad A\alpha x = \lambda \alpha x$$

$$\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \stackrel{!}{=} 5 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 3x_1 + 4x_2 \\ 2x_1 + x_2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 5x_1 \\ 5x_2 \end{pmatrix} \Rightarrow \begin{array}{l} 4x_2 = 2x_1 \\ 2x_1 = 4x_2 \end{array}$$

$$\Rightarrow x_1 = 2x_2$$

$$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad x = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 3x_1 + 4x_2 \\ 2x_1 + x_2 \end{pmatrix} \stackrel{!}{=} -1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$4x_2 = -4x_1$$
$$2x_1 = -2x_2$$

$$x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} -5 \\ 5 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} \rightarrow \det A = -5 \neq 0 \quad \text{invertible}$$

$$\det A = \lambda_1 \cdots \lambda_n$$
$$= -1 \cdot 5 = -5$$

$$A^{-1} = \frac{1}{-5} \begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix}$$