E600 Mathematics Fall Semester 2026

Problem Set 3

Analysis I: Multivariate Differentiation and Integration

Exercise 1: Important Objects of Differential Calculus

Consider $x_0 \in X \subseteq \mathbb{R}^n$, $f : X \mapsto \mathbb{R}$.
What is the type (e.g., real number, matrix, function, operator, etc.) of the following objects:

$$\frac{\partial f}{\partial x_1}(x_0), \quad \frac{df}{dx}, \quad \frac{\partial}{\partial x_1}, \quad \nabla f, \quad \frac{df(x_0)}{dx}$$

Exercise 2: Invertability

a.) Monotonic Functions and Injectivity

Show that any strictly monotonic function is injective.
Hint: It may be helpful to look up the formal definitions of strict monotonicity and injectivitiy.

b.) Some Examples

Determine which of the following functions are invertible, and if not, which criterion (injectivity or surjectivity) fails. In case of non-invertability, can you restrict the domain and/or codomain to achieve invertability?[1]

1. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto \cos(x)$
2. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto x^2$
3. $f : \mathbb{N} \mapsto \mathbb{N},\ n \mapsto n^4$
4. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto \mathbb{1}[x < 1](x-1)^2 + \mathbb{1}[x \geq 1]\log(x)$

Hint: for 4., investigate the two parts of the domain, $x < 1$ and $x \geq 1$, separately, and then think about whether putting the two parts together changes your conclusion. It may also be helpful to draw the function.

[1] You should understand the word "restrict" as cutting the set to the left/right, but do not manipulate it in the middle; e.g. if the initial set is $\mathbb{N}$, then $R = \mathbb{N} \cap [3, 100]$ is an allowed restriction, but $R = \{3, 9, 27, 87\}$ is not.

Exercise 3: Convexity

Investigate the following function with respect to (strict) convexity/concavity:

$$f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto ||x||$$

where $||\cdot||$ is a norm on $\mathbb{R}^n$, $n \in \mathbb{N}$.
Hint 1: Some universal properties of norms might be helpful.
Hint 2: A function is only both convex and concave if it is linear, i.e. $f(x+y) = f(x) + f(y)$ for *any* possible arguments x, y. Thus, once you showed that f satisfies one of the properties, you can check the other by investigating whether f is/can potentially be linear.

Exercise 4: Multivariate Differentiation

a.) Taylor Approximation: Univariate case

This exercise is meant as a fresh up for Taylor's theorem. Feel free to skip ahead to (b) *if you are already well-familiar with Taylor approximations.*

Compute the first and second order Taylor approximations to the exponential function $\exp : \mathbb{R} \mapsto \mathbb{R}_+, x \mapsto \exp(x)$ around $x_{0,1} = 1$. Evaluate both approximations at $x = -5$ and $x = 2$. Is one always better than the other? Are the approximations "good", i.e. are they close to the true value?

b.) Matrix Functions

Consider a matrix $A \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$.

(i) Show that $\frac{d}{dx}(Ax) = A$.

(ii) What is the derivative of $f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto x'Ax$?
Hint: Use *(i)* and the multivariate product rule.

(iii) If $A = \begin{pmatrix} 1 & \alpha \\ \beta & 4 \end{pmatrix}$, can you find values for α and β so that the second derivative of $x'Ax$ is positive definite everywhere? Can you find an alternative combination where A is positive semi-definite but not positive definite?

Exercise 5: Multivariate Differentiation

a.) Hessian Criterion for Convexity

Investigate the following function with respect to (strict) convexity/concavity:

$$f : \mathbb{R}^2 \mapsto \mathbb{R}, x = (x_1, x_2)' \mapsto \exp(x_1) + x_1 x_2 + 5x_1 + 4$$

Hint: Recall that we can use the second derivative to investigate convexity. The function is infinitely many times continuously partially differentiable, which can save you a few computational steps.

b.) Support Restriction?

Use the Hessian Criterion to investigate whether the function

$$f(x_1, x_2) = \frac{1}{2}(x_2^3 + 2x_1 x_2 + x_1^2)$$

is (strictly) convex or concave on $\mathbb{R}^3$, and otherwise try to find the support restrictions on which one of the properties holds.
Hint: The function is infinitely many times continuously partially differentiable, which can save you a few computational steps.

Exercise 6: Multivariate Integration

Consider an economy populated by a mass $[0,1]$ of firms that use capital k and labor l to produce output $y = f(k,l) = Ak^\alpha l^{1-\alpha}$, i.e. they use the same Cobb-Douglas production technology. Further, suppose that economy-wide output satisfies

$$Y = \int_{[0,1]\times[0,1]} f(k,l) d(k,l).$$

Amongst others, this relationship can be obtained from assuming that labor l and capital k are independently and uniformly distributed on $[0,1]$. However, it is not too important what this means here, it just ensures that the equality above holds.

Determine Y as a function of A and α. What do you conclude for the role of α, the relative importance of capital in the production process in terms of its relationship to Y?