Problem Set 3

Analysis I: Multivariate Differentiation and Integration

Exercise 1: Important Objects of Differential Calculus

Consider $x_0 \in X \subseteq \mathbb{R}^n$, $f : X \mapsto \mathbb{R}$.
What is the type (e.g., real number, matrix, function, operator, etc.) of the following objects:

$$\frac{\partial f}{\partial x_1}(x_0), \quad \frac{df}{dx}, \quad \frac{\partial}{\partial x_1} f, \quad \nabla f, \quad \frac{df(x_0)}{dx}$$

"∇f"

$\frac{df}{dx}(x_0)$

Value

$\frac{\partial f}{\partial x_1}(x_0) \in \mathbb{R}$

Function

$\frac{df}{dx}$, ∇f

Operator

$\frac{\partial}{\partial x_1} : D_1 \to F(\mathbb{R}^n, \mathbb{R})$

$\frac{\partial f}{\partial x}(x_0) \in \mathbb{R}^n$

$\mathbb{R} \to \mathbb{R}$

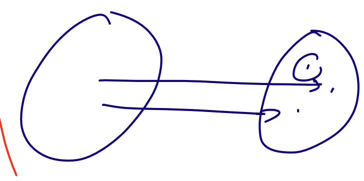
$x > y \Rightarrow f(x) > f(y)$ $<$

Exercise 2: Invertability

a.) Monotonic Functions and Injectivity

Show that any strictly monotonic function is injective.
Hint: It may be helpful to look up the formal definitions of strict monotonicity and injectivitiy.

Monotone $\Rightarrow$ injective



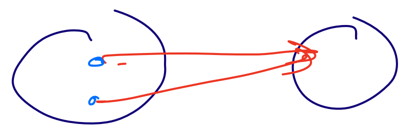
Injectivity

By contradiction

Absurd hypothesis

Assume $\exists\, x \neq y$ s.t. $f(x) = f(y)$

If $x \neq y \Rightarrow \boxed{x > y}$ or $y > x$



WLOG

Without loss of generality $x > y \overset{\text{mon.}}{\Longrightarrow} f(x) > f(y)$

$\Rightarrow f(x) \neq f(y)$

Contradiction $\perp$ ↯

injective ✓

b.) Some Examples

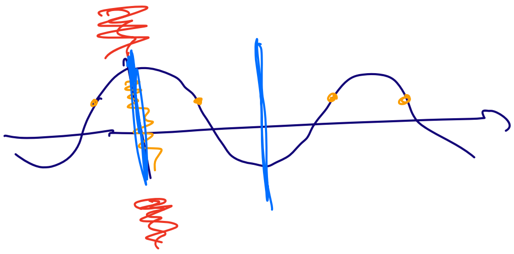
Determine which of the following functions are invertible, and if not, which criterion (injectivity or surjectivity) fails. In case of non-invertability, can you restrict the domain and/or codomain to achieve invertability?[1] $\leftrightarrow$ bijective function (one-to-one)

1. $f:\mathbb{R}\mapsto\mathbb{R},\ x\mapsto\cos(x)$
2. $f:\mathbb{R}\mapsto\mathbb{R},\ x\mapsto x^2$
3. $f:\mathbb{N}\mapsto\mathbb{N},\ n\mapsto n^4$
4. $f:\mathbb{R}\mapsto\mathbb{R},\ x\mapsto \mathbb{1}[x<1](x-1)^2+\mathbb{1}[x\geq 1]\log(x)$ $= \begin{cases} (x-1)^2 & x<1 \\ \log x & x\geq 1 \end{cases}$

Hint: for 4., investigate the two parts of the domain, $x<1$ and $x\geq 1$, separately, and then think about whether putting the two parts together changes your conclusion. It may also be helpful to draw the function.

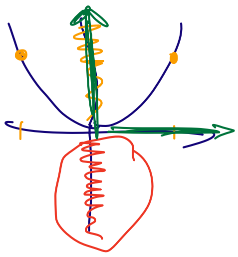
1\. $f: \mathbb{R} \to \mathbb{R} \quad x \to \cos x$

$f\ [0,\pi] \to [-1,1]$



2\. $f: \mathbb{R} \to \mathbb{R} \quad x \to x^2$

"$f^{-1} = \sqrt{x}$"

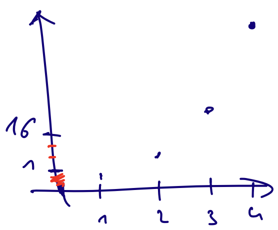


$f: \mathbb{R}^+_0 \to \mathbb{R}^+_0 : x \to x^2$

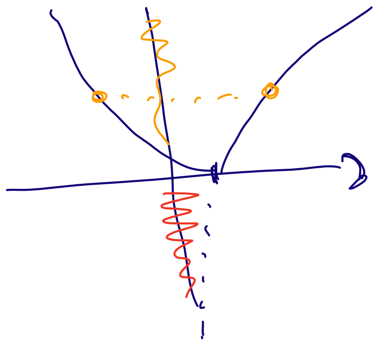
3\. $f: \mathbb{N} \to \mathbb{N} : n \to n^4$

$= \text{im} f = f(\mathbb{N})$

$f: \mathbb{N} \to \{n\in\mathbb{N} : \exists k\in\mathbb{N}\ n=k^4\}$



4\. $f: \mathbb{R}\to\mathbb{R} \quad x \to \begin{cases} (x-1)^2 & x<1 \\ \cos x & x\geq 2 \end{cases}$



$f: (-\infty,1) \to \mathbb{R}^+_0 \quad (x-1)^2$

$f: [1,\infty) \to \mathbb{R}^+_0 \quad \cos x$

[1] You should understand the word "restrict" as cutting the set to the left/right, but do not manipulate it in the middle; e.g. if the initial set is $\mathbb{N}$, then $R=\mathbb{N}\cap[3,100]$ is an allowed restriction, but $R=\{3,9,27,87\}$ is not.

Exercise 3: Convexity

Investigate the following function with respect to (strict) convexity/concavity:

$$f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto \|x\|$$

3. $\|a+b\| \leq \|a\| + \|b\|$
2. $\|\alpha a\| = |\alpha| \cdot \|a\|$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$, $n \in \mathbb{N}$.

Hint 1: Some universal properties of norms might be helpful.

Hint 2: A function is only both convex and concave if it is linear, i.e. $f(x+y) = f(x) + f(y)$ for *any* possible arguments x, y. Thus, once you showed that f satisfies one of the properties, you can check the other by investigating whether f is/can potentially be linear.

WTS: Condition convexity $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

$$\|\lambda x + (1-\lambda) y\| \leq \lambda \|x\| + (1-\lambda)\|y\|$$

Triangle ineq.

$$\|\underbrace{\lambda x}_{v_1} + \underbrace{(1-\lambda) y}_{v_2}\| \leq \|\lambda x\| + \|(1-\lambda) y\|$$

$$\overset{\text{Homog.}}{=} |\lambda| \, \|x\| + |1-\lambda| \, \|y\|$$

$$= \lambda \|x\| + (1-\lambda) \|y\|$$

$\longrightarrow$ (weak) convexity

str. convexity ✗ $\quad x = 0 \quad \|\lambda 0 + (1-\lambda) y\| = (1-\lambda)\|y\|$

$$= \lambda \cdot 0 + (1-\lambda)\|y\|$$

(weak) concavity ✗ $\quad x = -y \quad \lambda = \frac{1}{2}$

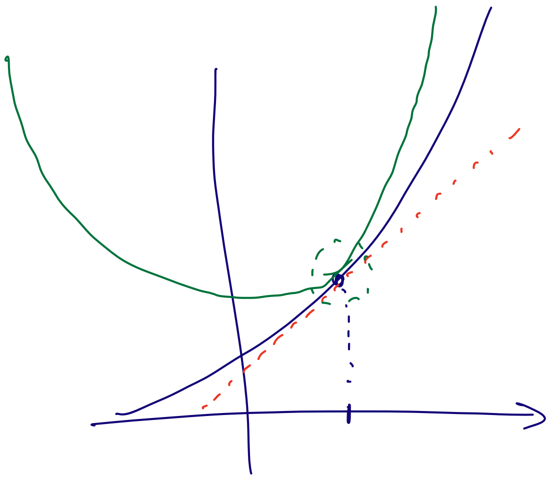
$$\|\lambda x + (1-\lambda) y\| = \|\tfrac{1}{2} x - \tfrac{1}{2} x\| = \|0\| < \tfrac{1}{2}\|x\| + \tfrac{1}{2}\|y\|$$

Exercise 4: Multivariate Differentiation

a.) Taylor Approximation: Univariate case

This exercise is meant as a fresh up for Taylor's theorem. Feel free to skip ahead to (b) *if you are already well-familiar with Taylor approximations.*

Compute the first and second order Taylor approximations to the exponential function $\exp : \mathbb{R} \mapsto \mathbb{R}_+, x \mapsto \exp(x)$ around $x_{0,1} = 1$. Evaluate both approximations at $x = -5$ and $x = 2$. Is one always better than the other? Are the approximations "good", i.e. are they close to the true value?



$$T_{1,x_0} = f(x_0) + f'(x_0)(x - x_0)$$
$$= e^{x_0} + e^{x_0}(x - x_0)$$

$$T_{2,x_0} = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$
$$= e^{x_0} + e^{x_0}(x - x_0) + \frac{1}{2} e^{x_0}(x - x_0)^2$$

$$f(x) = e^x \quad f' = f'' = f^{(n)} = e^x$$

$$T_{1,x_0}(2) = e + e(2-1) = 2e \qquad T_{1,x_0}(-5) = e + e(-5-1) = -5e$$

$$T_{2,x_0}(2) = e + e(2-1) + \frac{1}{2}e(2-1)^2 = \frac{5}{2}e \qquad T_{2,x_0}(-5) = e + e(-5-1) + \frac{1}{2}e(-5-1)^2$$
$$= 13e$$

$$e^2 \approx 2.718\ldots e \qquad e^{-5} \approx 0.0067$$

b.) Matrix Functions

Consider a matrix $A \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$.

(i) Show that $\frac{d}{dx}(Ax) = A$.

(ii) What is the derivative of $f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto x'Ax$?
Hint: Use *(i)* and the multivariate product rule.

(iii) If $A = \begin{pmatrix} 1 & \alpha \\ \beta & 4 \end{pmatrix}$, can you find values for α and β so that the second derivative of $x'Ax$ is positive definite everywhere? Can you find an alternative combination where A is positive semi-definite but not positive definite?

$$\left[\begin{pmatrix} 2 & \alpha+\beta \\ \alpha+\beta & 8 \end{pmatrix} x\right]'$$

$$=$$

$$x'Ax \to x'\begin{pmatrix} 2 & \alpha+\beta \\ \alpha+\beta & 8 \end{pmatrix}$$

$$\frac{d^2}{dx^2} = \begin{pmatrix} 2 & \alpha+\beta \\ \alpha+\beta & 8 \end{pmatrix} \quad \text{def } 2>0$$

$$\det = 2\cdot 8 - (\alpha+\beta)^2 > 0$$

$$(\alpha+\beta)^2 < 16 \quad |\alpha+\beta| < 4$$

(i) $$Ax = \begin{pmatrix} a_1^r \\ \vdots \\ a_n^r \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1^r \cdot x \\ \vdots \\ a_n^r\, x \end{pmatrix} \to J_{Ax} = \begin{pmatrix} \nabla a_1^r \cdot x \\ \vdots \\ \nabla a_n^r\, x \end{pmatrix}$$

$$\nabla a_i^r \cdot x = \nabla\left(a_{i1}x_1 + a_{i2}x_2 + \dots a_{in}x_n\right)$$

$$\nabla = [a_{i1} \quad a_{i2} \quad \dots \quad a_{in}]$$

$$J_{Ax} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = A$$

ii) $$\frac{d}{dx}\, x'Ax \qquad f(x) = ax^2 \qquad \frac{d}{dx} f^T g = f^T \frac{d}{dx} g + g^T \frac{d}{dx} f$$

$$= x' I_n' A x \Rightarrow \frac{d}{dx} (I_n x)'(Ax)$$

$$= (I_n x)' A + (Ax)' I_n$$

$$= x' I_n A + x' A' I_n = x'A + x'A'$$

$$= x'(A + A')$$

Symmetric A $\quad x'(A+A') = 2x'A \qquad f'(x) = 2ax$

Exercise 5: Multivariate Differentiation

a.) Hessian Criterion for Convexity

Investigate the following function with respect to (strict) convexity/concavity:

$$f : \mathbb{R}^2 \mapsto \mathbb{R}, x = (x_1, x_2)' \mapsto \exp(x_1) + x_1 x_2 + 5x_1 + 4$$

Hint: Recall that we can use the second derivative to investigate convexity. The function is infinitely many times continuously partially differentiable, which can save you a few computational steps.

$$f(x_1, x_2) = e^{x_1} + x_1 x_2 + 5x_1 + 4$$

$$\nabla_f = \left(e^{x_1} + x_2 + 5 \quad x_1 \right)$$

$$\det = e^{x_1} \cdot 0 - 1 = -1 < 0$$

$$H_f = \begin{pmatrix} \boxed{e^{x_1}}^{>0} & 1 \\ 1 & 0 \end{pmatrix} \text{ not pos. or neg. definite}$$

b.) Support Restriction?

Use the Hessian Criterion to investigate whether the function

$$f(x_1, x_2) = \frac{1}{2}(x_2^3 + 2x_1x_2 + x_1^2)$$

is (strictly) convex or concave on $\mathbb{R}^3$, and otherwise try to find the support restrictions on which one of the properties holds.

Hint: The function is infinitely many times continuously partially differentiable, which can save you a few computational steps.

$$f(x_1, x_2) = \frac{1}{2}\left(x_1^2 + 2x_1x_2 + x_2^3\right)$$

$$D_f = \left(x_1 + x_2 \quad x_1 + \frac{3}{2}x_2^2 \right)$$

det > 0

$$\det H_f = \det \begin{pmatrix} 1 & 1 \\ 1 & 3x_2 \end{pmatrix} = 3x_2 - 1 \geq 0$$

$$3x_2 - 1 > 0$$

$$3x_2 > 1 \iff x_2 > \frac{1}{3}$$

$$f: \mathbb{R} \times \left(\tfrac{1}{3}, \infty\right) \to \mathbb{R} \quad \text{str. convex}$$

Exercise 6: Multivariate Integration

Consider an economy populated by a mass $[0,1]$ of firms that use capital k and labor l to produce output $y = f(k,l) = Ak^{\alpha}l^{1-\alpha}$, i.e. they use the same Cobb-Douglas production technology. Further, suppose that economy-wide output satisfies

$$Y = \int_{[0,1]\times[0,1]} f(k,l)d(k,l).$$

Amongst others, this relationship can be obtained from assuming that labor l and capital k are independently and uniformly distributed on $[0,1]$. However, it is not too important what this means here, it just ensures that the equality above holds.

Determine Y as a function of A and α. What do you conclude for the role of α, the relative importance of capital in the production process in terms of its relationship to Y?

$$f(k,l) = Ak^{\alpha}l^{1-\alpha}$$

$$\int_{[0,1]\times[0,1]} Ak^{\alpha}l^{1-\alpha}\,d(k,l)$$

$$\int_{[0,1]} \overbrace{\int_{[0,1]} Ak^{\alpha}l^{1-\alpha}\,dk}^{Al^{1-\alpha}\int k^{\alpha}dk}\,dl$$

$$Al^{1-\alpha}\frac{1}{\alpha+1}$$

$$A\frac{1}{\alpha+1}\int_0^1 l^{1-\alpha}\,dl$$

$$A\frac{1}{\alpha+1}\frac{1}{2-\alpha}$$

$$f(x) = g(x_1 \dots x_k)\,h(x_{k+1}\dots x_n) \qquad \int g \cdot \int h$$

$$A\int_0^1 k^{\alpha}\,dk \int_0^1 l^{1-\alpha}\,dl = A\left[\frac{1}{\alpha+1}k^{\alpha+1}\right]_0^1 \left[\frac{1}{2-\alpha}l^{2-\alpha}\right]_0^1$$

$$= A\frac{1}{\alpha+1}\cdot\frac{1}{2-\alpha} = \frac{A}{(\alpha+1)(2-\alpha)} = \frac{A}{2-\alpha^2+\alpha}$$