# E600 Mathematics

Fall Semester 2026

## Problem Set 4

*Analysis II: Optimization*

### Exercise 1: Optimization Basics

#### a.) Concepts

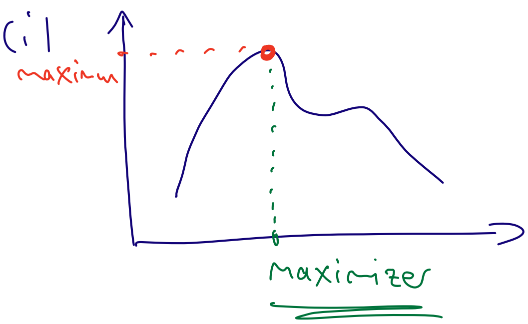
*(i)* Verbally explain the difference and relationship between a maximum and a maximizer.

*(ii)* Let $x_1^*, x_2^* \in X$ such that $x_1^* \in \arg\max_{x \in X} f(x)$ and $x_2^* \in \arg\max_{x \in C} f|_C(x)$ where $C \subseteq X$ is a constraint set. Can you have $f(x_2^*) > f(x_1^*)$? Explain why (not).

maximizer

*(iii)* Can be $\arg\max_{x \in X} f(x)$ empty? Can it have more than one argument if there is a strict global maximizer?

(i)



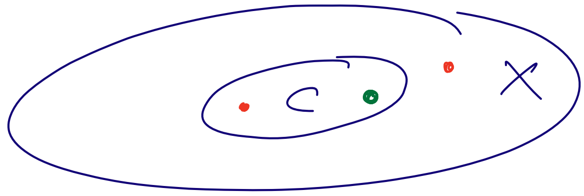

utility ordinal

cost cardinal

(ii) $x_1 \geq 0$ $x_1 + x_2 = \text{const.}$ $\rightarrow$ $X \rightarrow C$

binding

$f_1(x^*) > f_2(x^*)$




$f_1(x^*) = f_2(x^*)$

slack

(iii) a) $\arg\max f = \emptyset$ $\rightarrow$ $\max_{\mathbb{R}} x^2$

b) strict global maximum $\Rightarrow$ $|\arg\max f| = 1$

e.g. concave function

→ cont. f over compact domain
↳ closed
↳ bounded

## b.) Continuity and Weierstrass

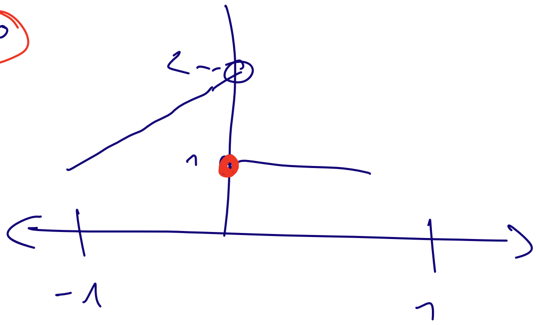
always exist max & min

Give an example of a discontinuous function on a compact domain that does not have a global maximum.

*Hint:* You may want to define a "split function", i.e. $f(x) = \begin{cases} f_1(x) & \text{if } \ldots \\ f_2(x) & \text{else} \end{cases}$.

$X = [-1, 1]$

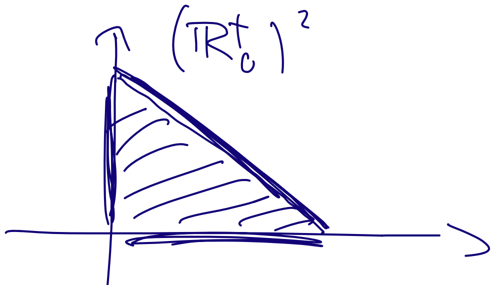
$f = \begin{cases} x+2 & x<0 \\ 1 & x \geq 0 \end{cases}$




→ no maximum exists.

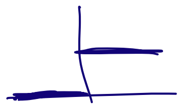
⇒ supremum $= 2$
infimum

$1 \cdot x_1 + 2 \cdot x_2 \leq 5$




### c.) Weierstrass: Concrete Examples

Does the Weierstrass Extreme Value Theorem apply to the functions



1. $f: \mathbb{R} \mapsto \mathbb{R},\ x \mapsto x^3$
2. $f: (0, \pi) \mapsto \mathbb{R},\ x \mapsto \cos(x)$
3. $f: \{x \in \mathbb{R}^2_+ : \overset{p}{(1,2)} \cdot x \leq \overset{y}{5}\} \mapsto \mathbb{R},\ x \mapsto x_1 + x_2$
4. $f: [-1, 1] \mapsto \mathbb{R},\ x \mapsto \mathbb{1}[x > 0]$
5. $f: [0, \pi] \mapsto \mathbb{R},\ x \mapsto (\cos(x) + 2)^{\sin(x)}$
6. $f: \bar{B}_1(\mathbf{0}) \mapsto \mathbb{R},\ x \mapsto x'x$ where $\mathbf{0} \in \mathbb{R}^5$

Is there an example in these functions that demonstrates that the Weierstrass Extreme Value Theorem only formulates only a sufficient, but not an equivalent condition for existence of the global extreme values?

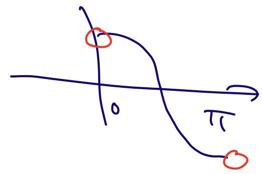
*Hint:* If you get stuck on 3., ask yourself whether the domain resembles a set that you know from economics.

a) Continuity $f$

b) Closedness $X$

c) Boundedness $X$

} Compact $X$




1.) Continuity ✓ $\mathbb{R}$ closed ✓ $\mathbb{R}$ unbounded X

2.) Cont. ✓ $(0, \pi)$ open X $(0, \pi)$ bounded ✓

3.) Cont. ✓ BS closed ✓ BS bounded ✓

-> max, min exist.

4.) Cont. X $[-1, 1]$ closed ✓ $[-1, 1]$ bounded ✓

5.) Cont. ✓ $[0, \pi]$ closed ✓ $[0, \pi]$ bounded ✓

6.) Cont. ✓ $\bar{B}_1(0)$ closed ✓ $\bar{B}_1(0)$ bounded ✓

Cobb-Douglas: $u = x_1^{\alpha} x_2^{\beta}$ $\qquad Y = p_1 x_1 + p_2 x_2$

$x_1^*, x_2^*$ $\qquad \hookrightarrow v = \alpha \ln x_1 + \beta \ln x_2$ $\qquad p_1 x_1 + p_2 x_2 - Y = 0$

$$\mathcal{L} = \alpha \ln x_1 + \beta \ln x_2 - \lambda (p_1 x_1 + p_2 x_2 - Y)$$

$D\mathcal{L} \overset{!}{=} 0$

$$D\mathcal{L}' = \begin{pmatrix} \frac{\alpha}{x_1} - \lambda p_1 \\ \frac{\beta}{x_2} - \lambda p_2 \\ p_1 x_1 + p_2 x_2 - Y \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{aligned} \frac{\alpha}{x_1} &= \lambda p_1 \\ \frac{\beta}{x_2} &= \lambda p_2 \\ p_1 x_1 + p_2 x_2 &= Y \end{aligned}$$

$$\frac{\partial/\partial x_1}{\partial/\partial x_2}: \quad \frac{\alpha}{\beta} \frac{x_2}{x_1} = \frac{p_1}{p_2} \iff p_1 x_1 = \frac{\alpha}{\beta} p_2 x_2 \quad \text{insert in } \frac{\partial}{\partial \lambda}$$

$$Y = \frac{\alpha}{\beta} p_2 x_2 + p_2 x_2 = \frac{\alpha + \beta}{\beta} p_2 x_2 \Rightarrow \boxed{\begin{aligned} x_2^* &= \frac{\beta}{\alpha+\beta} \frac{Y}{p_2} \\ x_1^* &= \frac{\alpha}{\alpha+\beta} \frac{Y}{p_1} \end{aligned}}$$

## Exercise 2: Economic Formulation and Application

*Disclaimer:* This problem is pretty long. It covers examples for many of economics' most popular optimization problems, including utility maximization, cost minimization, computation of indirect utility functions in a model parameter, welfare maximization and an exchange economy problem, which are all important in their own right, and you should have seen them at least once.

Suppose we have two individuals, Anna and Ben. Both like to spend their free time performing only two activities: relaxing ($r$) and going climbing ($c$). Otherwise, they don't derive utility from any other source. Suppose that an hour of relaxation costs 1 Euro (e.g. for a Netflix account, food and drinks, or whatever you like to consume in your free time), and an hour of climbing costs 10 Euro(equipment, gym subscription, etc.). Suppose that both Anna and Ben are employed at the same job, and can work for a net hourly wage of 15 Euro to generate income; they both have no initial wealth. Their preferences differ: we have

$$u_A(r,c) = r^{\frac{1}{2}}c^{\frac{1}{2}} \quad \text{and} \quad u_B(r,c) = r^{\frac{1}{3}}c^{\frac{2}{3}}$$

This means that Ben puts more weight on climbing whereas both activities are equally weighted for Anna.

h: working hours

### (i) Writing Down and Simplifying the Problem

Formulate the problem that Anna faces when maximizing utility within a given day that has 24 hours. Simplify the problem as much as possible.

### (ii) Budget Constraint Interpretation

How can you interpret the budget constraint that you obtain after simplification?

### (iii) Solving Anna's Problem

Solve Anna's utility maximization problem (finding the optimal distribution of time across activities is enough; the value of utility does not matter).

$\max u_A = r^{1/2} c^{1/2}$

$24 = h + r + c \rightarrow h = 24 - r - c$

$15 \cdot h = 10c + 1r$

$\rightarrow \tilde{u}_A = rc$

$15(24 - r - c) \Rightarrow \boxed{360 = 25c + 16r}$

$\max r \cdot c \quad \text{s.t.} \quad 25c + 16r - 360 = 0$

$r^* = \frac{1}{2} \cdot \frac{360}{16} = \frac{360}{32} \qquad c^* = \frac{1}{2} \cdot \frac{360}{25} = \frac{360}{50}$

$\Rightarrow u_A^* = \sqrt{\frac{360}{32} \cdot \frac{360}{50}} = \sqrt{81} = 9$

$u^* = 10$

$\max \; u \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 = c = c^*$

$\min \; c \quad \text{s.t.} \quad u(x_1, x_2) = \bar{u} = 10$

$c^*$

25 vs 16

10 vs 1

### (iv) A Problem with Savings

Assume now that Anna has some savings and does not need to work on the day we consider in our optimization problem here. ~~Given her utility function, what is the~~ minimum amount of money that she needs to spend to achieve at least the same level of utility as before? (Even though she does not need to work here, she can still not spend more than 24 hours on both activities combined.)

*Hint:* Don't worry if your results don't give nice numbers anymore, you will need a calculator for this exercise. You may round all intermediate results to two digits.

### (v) Differences in Results

How do you explain the difference in results of *(iv)* and *(iii)*, especially in terms of the level of expenditures?

$$\min \; 10c + 1r \qquad \sqrt{rc} \overset{u_A^*}{=} 9 \Leftrightarrow rc = 81 \Leftrightarrow r = \frac{81}{c}$$

$$r + c \leq 24 \rightarrow \text{Karush-Kuhn-Tucker}$$

$$\min \; 10c + \frac{81}{c} \qquad \text{s.t.} \; \frac{81}{c} + c - 24 \leq 0$$

1) $\mu = 0 \qquad 10 - \frac{81}{c^2} \overset{!}{=} 0 \Rightarrow c^2 = \frac{81}{10} \Rightarrow c^* = \frac{9}{\sqrt{10}}, \; r^* = 9 \cdot \sqrt{10} > 27 > 24$

Contradiction

2) $\mu \neq 0 \iff r + c = 24 \rightarrow \frac{81}{c} + c = 24$

$$c^2 - 24c + 81 = 0$$

$$c_{1/2} = \frac{24}{2} \pm \sqrt{(12)^2 - 81} = 12 \pm \sqrt{63} \Rightarrow c^* \approx 19.94, \quad r^* \approx 4.06$$

$u_B = r^{1/3} c^{2/3} \rightarrow \tilde{u}_B = r c^2$

### (vi) Ben's Wage Problem

$\rightarrow u_A^* = 9$

How much would Ben need to earn per hour to afford Anna's level of utility if he has no savings? You may not be able to solve for the wage to the cent; thus, assume that the wage is an integer value. You can use without proof that utility is strictly increasing in the wage and check in steps of 1 whether a given wage yields at least the desired level of utility.
*Hint:* Solve Ben's utility maximization problem in analogy to Anna's with variable wage to derive the maximal utility as a function of the wage in a first step.

### (vii) Ben's Bound on Utility

What level of utility can Ben maximally attain as his wage increases? Why is utility not unbounded above? What can you say about Ben's utility and time allocation when he does not have to work from this investigation?

$\max \; r \cdot c^2 \quad \text{s.t.} \quad w \cdot 24 = (w+10)c + (w+1)r$

$c^* = \frac{2}{3} \frac{24 w}{w+10} \qquad r^* = \frac{1}{3} \frac{24 w}{w+1} \qquad r^*(w),\; c^*(w)$

$\rightarrow$ Indirect utility function $\qquad \lim f = f(\lim)$

$V(w) = r^*(w)^{1/3} c^*(w)^{2/3}$

$w = 15 \rightarrow V(w) \approx 8.84 < 9 = u_A^*$

$w = 16 \rightarrow V(w) \approx 9.004 > 9 = u_A^*$

$$\lim_{w \to \infty} V(w) = \lim \left[ \left(8 \frac{w}{w+1}\right)^{1/3} \left(16 \frac{w}{w+10}\right)^{2/3} \right]$$

$$= \lim \left(8 \frac{w}{w+1}\right)^{1/3} \cdot \lim \left(16 \frac{w}{w+10}\right)^{2/3}$$

$$= \left(8 \cdot \lim \left(\frac{w}{w+1}\right)\right)^{1/3} \cdot \left(16 \lim \frac{w}{w+10}\right)^{2/3}$$

$$= 8^{1/3} \cdot 16^{2/3} < \infty$$

$\rightarrow$ Saving's problem

### (viii) Welfare Maximization with Tradeable Goods: Problem

Consider the problem where now, $c$ and $r$ are tradeable goods, e.g. cookies and rice (in kg). Suppose that Anna and Ben live on a deserted island, on which there are 10 boxes of cookies and 12 kg of rice. Formulate the welfare-maximizing resource allocation problem when equal weight is given to both individuals and simplify it as much as possible. How do you proceed to find the solution (verbally)?

### (ix) Welfare Maximization with Tradeable Goods: Solution

What is the allocation that solves the problem of *(viii)*?
*Hint:* Beware of border solutions.

$r_A + r_B = 12 \qquad c_A + c_B = 10 \qquad W = U_A + U_B$

$\max \; r_A^{1/2} c_A^{1/2} + r_B^{1/3} c_B^{2/3}$

$r_A + r_B - 12 = 0 \qquad c_A + c_B - 10 = 0$

$r_A, r_B, c_A, c_B \geq 0$

$x_A = x_B = 0 \quad \times$

if $r_i = 0 \Rightarrow c_i = 0$

$c_i = 0 \Rightarrow r_i = 0$

1) internal solution $\rightarrow \mu = 0$

2) $r_A = 12, \; c_A = 10 \rightarrow U_A = 10.95$

3) $r_B = 12, \; c_B = 10 \rightarrow U_B = 10.63$

Solve 1) $\mathcal{L} = r_A^{1/2} c_A^{1/2} + r_B^{1/3} c_B^{2/3} - \lambda_1 (r_A + r_B - 12) - \lambda_2 (c_A + c_B - 10)$

$$\nabla \mathcal{L}' = \begin{pmatrix} \frac{1}{2} r_A^{-1/2} c_A^{1/2} - \lambda_1 \\ \frac{1}{2} r_A^{1/2} c_A^{-1/2} - \lambda_2 \\ 1/3\, r_B^{-2/3} c_B^{2/3} - \lambda_1 \\ 2/3\, r_B^{1/3} c_B^{-1/3} - \lambda_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}$$

$\frac{I}{III} \quad \frac{1}{2} \left(\frac{c_A}{r_A}\right)^{1/2} = \frac{1}{3} \left(\frac{c_B}{r_B}\right)^{2/3}$

$\frac{II}{IV} \quad \frac{1}{2} \left(\frac{r_A}{c_A}\right)^{1/2} = \frac{2}{3} \left(\frac{r_B}{c_B}\right)^{1/3}$

$r_A + r_B = 12$

$c_A + c_B = 10$

$\rightarrow c_A = 7.09 \qquad c_B = 2.91 \qquad \rightarrow \omega = 10.99$

$r_A = 9.96 \qquad r_B = 2.04$

### (x) Exchange Economy

Finally, consider again the island scenario but now assume that Anna initially owns all the cookies and Ben owns all the rice. Assume that they can freely discuss exchanging the goods, have perfect information about all aspects of the trade and there is no asymmetry in terms of negation power between the two. At what ratio do they trade the goods, and what is the final allocation? How does aggregate welfare compare to the previous exercise?
*Hint:* You can express both agents' utility to trading a given quantity of goods at a fixed ratio as a function of the ratio and the goods quantity, and then solve for concrete values that sustain an equilibrium by thinking about the condition that ensures both agents do not want do deviate.

$c_A = 10$ $\quad r_B = 12$ $\quad p_r, p_c \rightarrow q = \frac{p_c}{p_r}$

A: $\max rc$ s.t. $10 p_c = c_A p_c + r_A p_r$

$10 q = c_A q + r_A$

$r_A^* = \frac{1}{2}\frac{10q}{1} = 5q$

$c_A^* = \frac{1}{2}\frac{10q}{q} = 5$

$r_A^*(q),\ c_A^*(q)$

B: $\max rc^2$ s.t. $12 p_r = c_B p_c + r_B p_r$

$12 = c_B q + r_B$

$r_B^* = \frac{1}{3}\frac{12}{1} = 4$

$c_B^* = \frac{2}{3}\frac{12}{q} = \frac{8}{q}$

$r_B^*(q),\ c_B^*(q)$

choose $q$ s.t.

Market clearing conditions

$$c_A^*(q) + c_B^*(q) = 10$$

$$r_A^*(q) + r_B^*(q) = 12$$

$$r_A^* + r_B^* = 5q + 4 = 12 \Rightarrow 5q = 8 \quad \boxed{q = \frac{8}{5}}$$

$$c_A^*\left(\tfrac{8}{5}\right) + c_B^*\left(\tfrac{8}{5}\right) = 5 + \frac{8}{\frac{8}{5}} = 5 + 5 = 10 \ \checkmark$$

$p_c = 8$, $p_r = 5$ $\quad p_c = \frac{8}{5} \cdot p_r$

Clearing all but one market clears the last remaining one