E600 Mathematics Fall Semester 2026

Problem Set 5

Statistics: Probability Theory, Random Variables

Exercise 1: Bayes' Rule and Probabilities

a.) Conditional Probability

Given two events A and B, verbally state what the following objects mean?
$P(A)$, $P(A \cup B)$, $P(A \cap B)$, $P(A|B)$. What is the definition of $P(A|B)$ in terms of the other objects?

b.) Bayes' Rule

Apply Bayes' rule to solve the following problems:

(i) In an economy, $\frac{1}{3}$ of all firms shut down, after a new regulation is introduced that affects $\frac{1}{3}$ of all firms. Among the firms shutting down, $\frac{2}{3}$ were regulated. What is the probability that a regulated firm had to shut down? Is this probability higher than the probability that an unregulated firm has to shut down?

(ii) Suppose 10 % of the male labor force and 5 % of the female labor force are unemployed. Suppose that the labor force consists to 30 % of males and the remaining 70 % are female. What is the probability that a randomly drawn member of the labor force will be unemployed?

Exercise 2: Expected Value, Variance and CDF

Take a random variable X with the following density as given:

$$f_X(x) = \frac{1}{4} I_{[0,4]}(x)$$

$I_{[0,4]}(x)$ is an indicator function that returns the value 1 if $x \in [0,4]$ and zero otherwise.

(i) Derive the cumulative distribution function $F_X(t)$ for any real number t.
(ii) Calculate the expected value of X. Use the definition of the expected value from chapter 5.
(iii) What is the variance of X?

Exercise 3: Properties of Variance and Covariance

Take RVs X, Y, Z as given. Take $a, b \in \mathbb{R}$. Recall these rules for the expected values:

- $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$ (Linearity)
- For two independent random variables $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

a.) Properties of the Variance

Using the above properties of expected values, show the following:
(i) $Var(aX + b) = a^2 Var(X)$
(ii) for independent X, Y: $Var(X + Y) = Var(X) + Var(Y)$.

b.) Properties of the Covariance

Using the above properties of expected values, show the following:
(i) $Cov(aX + b, Y) = aCov(X, Y)$
(ii) $Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)$
(iii) for independent X, Y: $Cov(X, Y) = 0$

Exercise 4: Convergence of Random Variables

A good place to start working with convergence theorems for random variables is applying them to show that an estimator is consistent, i.e., that an estimator converges in probability to the object we intend to estimate. While we did not introduce the Ordinary Least Squares Estimator in this years course, we can already show that it is consistent, i.e. that it converges to the parameter we wish to identify.

Use the convergence theorems from chapter 5 to show that the OLS estimator $\hat{\beta}$ is consistent, i.e. that:

$$\Big(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\Big)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i y_i \xrightarrow{p} \beta = \mathbb{E}(X_i X_i')^{-1}\mathbb{E}(X_i y_i)$$

where X_i is a iid random vector.

Hint: Show convergence in probability for the two averages.
Slutsky's theorem (which will be introduced in E603) implies that for two series of random variables X_n, Y_n such that $X_n \xrightarrow{p} C$ and $Y_n \xrightarrow{p} T$, if $|C|, |T| < \infty$, then $X_n Y_n \xrightarrow{p} CT$.